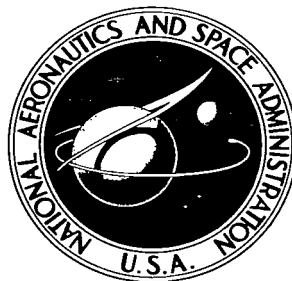


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**HEAT TRANSFER TO LIQUID-METAL FLOW  
IN A ROUND TUBE OR FLAT DUCT WITH  
HEAT SOURCES IN THE FLUID STREAM**

*by Robert M. Inman*

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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# HEAT TRANSFER TO LIQUID-METAL FLOW IN A ROUND TUBE OR FLAT DUCT WITH HEAT SOURCES IN THE FLUID STREAM

by Robert M. Inman  
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## SUMMARY

An analysis was carried out to determine the forced-convection heat-transfer characteristics for flow of a liquid metal in a flow passage with heat sources in the fluid stream. Two geometries commonly encountered in practice were selected for analysis: the circular tube and the parallel-plate channel, or flat duct. The study was based on the assumptions of (1) a fully developed velocity profile that is uniform over the passage cross section (slug flow), (2) molecular conductivity being appreciably larger than eddy conductivity, and (3) spatially uniform internal heat generation. The model is a fairly accurate representation for the turbulent flow of liquid metals for Péclet numbers less than 100. The results obtained apply along the entire length of the particular conduit, that is, in the thermal entrance region as well as in the region where the temperature profiles are fully developed. The eigenvalues and constants required for calculating temperature distributions and heat-transfer characteristics are determined analytically. The effects of the ratios that determine the relative role of internal heat generation to that of wall heat transfer or temperature driving force on the heat-transfer characteristics are investigated. Numerical results for wall temperatures, wall heat fluxes, and Nusselt numbers are presented graphically for a number of thermal boundary conditions that are of practical engineering interest.

## INTRODUCTION

Because of wide technical application, the flow and use as a heat-transfer medium of liquid metals in pipes and channels has been an active subject of investigation for a number of years. Advances in modern technology and the successful analysis and design of advanced liquid-metal systems for space propulsion and power generation are obliging the present day designer to understand the forced-convection heat-transfer character-

istics of laminar or turbulent liquid-metal flow in circular and noncircular ducts with heat generation in the fluid stream. This class of heat transfer in liquid-metal duct flow has many applications - for example, liquid-metal-fuel reactors (refs. 1 to 3), electromagnetic pumps and flowmeters (refs. 4 to 7), equipment for the electrolytic removal of gases from liquid metals (ref. 8), and liquid-metal magnetohydrodynamic power generators (refs. 9 to 12). The flow in these devices will be accompanied by wall-heating conditions and by heat generated internally through radioactive fission products or through viscous or Joule heating. A factor of importance for the proper operation of these devices is maintaining a satisfactory temperature distribution along the duct walls. The designer, therefore, must be able to compute the temperature distribution along the walls under these conditions and to know how much heat must be removed to cool the walls and thus to prevent temperatures from exceeding design limits. The ducts commonly employed in these devices are the circular tube and the flat duct. Turbulent flow is frequently encountered in actual operation, and it is this regime that is of interest here.

Some aspects of the general problem of turbulent forced-convection liquid-metal heat transfer in tubes and channels with heat generation in fluid stream have received a little experimental and theoretical work.

In references 13 and 14, turbulent heat transfer in a tube with liberation of heat in the flow is studied experimentally. The data, however, are for the regions where the temperature distribution is fully developed, and the results do not apply in the thermal entrance region. The study was further restricted by considering an insulated tube so that the tube wall was adiabatic. The effects of a magnetic field on turbulent forced-convection heat transfer in a tube has been investigated experimentally in references 15 and 16; however, the results apply for the condition of fully established temperature profiles beyond the thermal entrance region.

In reference 17, an analytical study has been made of the problem of heat transfer to a fluid flowing in a tube with internal heat sources and wall heat transfer. The fluid had a range of Prandtl number from 0.7 to 100. These results, however, are not generally applicable to liquid metals whose high thermal conductivities give a range of Prandtl numbers from approximately 0.001 to 0.1.

Heat transfer to liquid metals flowing turbulently between parallel plates has been analytically investigated in references 18 to 21. The results in references 18 and 19 are for the conditions of a power-law velocity profile, uniform wall temperatures, and no internal heat generation. The results in reference 20 are for the conditions of uniform wall heat fluxes and fully established temperature profiles; the effect of internal heat generation has not been considered. The effect of different wall temperatures and viscous dissipation on the heat transfer in the thermal entrance region of turbulent flows between

parallel plates was studied in reference 21 for a few combinations of Reynolds and Prandtl moduli.

The convective heat transfer for a liquid metal flowing turbulently through a circular or a noncircular duct has been studied analytically through slug-flow conduction solutions in references 22 and 23. The results are, however, for the regions where the temperature distribution is fully developed. The analyses were further restricted by not considering internal heat generation in the fluid.

The purpose of the present study is to obtain an understanding of the steady forced-convection heat transfer to liquid-metal flow in round tubes and in parallel-plate channels (often referred to as flat ducts) with heat sources in the fluid stream. The walls of the conduits are postulated to have a uniform heat flux flowing through them, or else are assumed maintained at a constant temperature, and the temperature distribution along the walls, or the heat flux variation along the conduit length required to maintain the wall temperature constant, is to be determined.

Particular consideration will be given to a simplified model; this physical model leads to tractable mathematical problems and yet contains many of the important features of heat transfer to turbulent liquid-metal duct flow. This model, therefore, should provide preliminary heat-transfer design results and be useful in gaining an understanding of more complex problems.

The idealized system that defines heat transfer in a liquid metal flowing turbulently in a duct and with internal heat generation in the fluid is based on the following assumptions:

(1) The turbulent velocity profile is fully established and is represented by a uniform distribution across the passage cross section (often referred to as slug flow).

(2) The eddy diffusivity of heat is small compared with the normal thermal diffusivity and can be neglected.

(3) Longitudinal heat conduction is small compared with convection and transverse heat conduction and can be neglected.

(4) The internal heat generation is spatially uniform.

(5) The fluid properties are invariant with temperature.

(6) Steady state exists.

Reference 22 points out that the slug-flow velocity profile is a good approximation for a turbulent liquid-metal system when the product of Reynolds and Prandtl moduli (termed the Péclet number,  $P_e$ ) is less than 100. References 24 and 25 indicate that, for liquid-metal heat transfer, the molecular heat-transfer term will be important at low Reynolds moduli in comparison with the turbulent term. The third assumption has been shown in reference 26 to introduce a negligible error for Péclet moduli equal to or greater than approximately 100, which is not consistent with the slug-flow approximation requirement.

Nevertheless, the slug-flow model is retained because of the resulting mathematical simplicity.

References 22 and 23 suggest that the slug-flow problem does reveal the essential physical behavior of the turbulent liquid-metal system at low Prandtl and Reynolds moduli, although the numerical magnitudes in the results for heat transfer are somewhat in error. The advantage of using this simplification is that exact mathematical solutions can be obtained for various thermal boundary conditions, and hence, the combined influence of internal heat generation and the wall-heating conditions can be demonstrated.

In the present investigation, numerical results are provided for the case of internal heat sources that are uniform across the duct cross section and along the duct length. The results, however, can be extended to include sources that vary in the transverse and longitudinal directions by application of the procedure given in reference 17. The results obtained apply along the entire length of the particular conduit, that is, in the thermal entrance region as well as in the region where the temperature profiles are fully developed. Numerical results are evaluated for a number of thermal boundary conditions that are of practical interest.

The present study is divided into two major sections, each dealing with a particular duct. The first section will consider liquid-metal heat transfer in a circular tube, since this is a flow passage commonly encountered in practical applications. The next section will deal with liquid-metal heat transfer in a parallel-plate channel. Flow between parallel plates may be expected to approximate (1) the flow through annuli where the radius ratio is close to unity, and (2) the flow through a rectangular duct if one side of the rectangle is large compared with the other.

## HEAT TRANSFER TO A LIQUID METAL FLOWING IN A ROUND TUBE

This section of the investigation will be subdivided into two subsections dealing, respectively, with uniform internal heat generation in the presence of a uniform wall heat flux and with uniform internal heat generation in the presence of a uniform wall temperature.

### Uniform Heat Generation and Uniform Wall Heat Flux

The geometry considered in the present development is shown in figure 1. The liquid metal flows from left to right. The part of the tube to the left of  $x = 0$  is a hydrodynamic starting section used to produce a fully established turbulent velocity profile at  $x = 0$ . The fluid temperature at the tube entrance is uniform across the section at a value  $t_e$ .

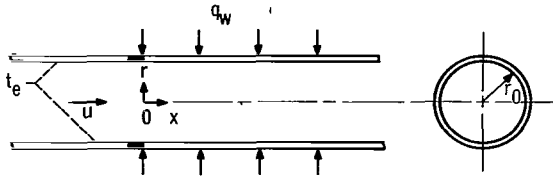


Figure 1. - Physical model and coordinate system for circular tube with wall heat transfer.

In the section of the tube to the right of  $x = 0$ , a heating process takes place that includes a uniform internal heat generation in the liquid metal and a uniform heat transfer at the tube wall.

The established turbulent velocity profile is approximated by the uniform distribution

$$u = U \quad (1)$$

(Symbols are defined in the appendix.) The differential equation describing convective heat transfer for the idealized system considered earlier is

$$u \frac{\partial t}{\partial x} = \frac{\kappa}{\rho c_p} \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial t}{\partial r} \right) + \frac{Q}{\rho c_p} \quad (2)$$

The linearity of the energy equation enables the temperature distribution  $t(x, r)$  to be written as the sum of two parts:

$$t(x, r) = t_Q(x, r) + t_q(x, r) \quad (3)$$

in which the temperature  $t_Q$  corresponds to the situation where there is a uniform internal heat generation in the fluid flowing through a tube with insulated walls ( $q_w = 0$ ), and the temperature  $t_q$  corresponds to the situation where there is a uniform heat transfer  $q_w$  at the tube wall but no internal heat generation ( $Q = 0$ ). The solution to the general problem is then given by equation (3).

The dimensionless equations and boundary conditions used to determine  $t_Q(x, r)$  and  $t_q(x, r)$  are, respectively,

$$\frac{\partial t_Q}{\partial \xi} = \frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} \left( \bar{r} \frac{\partial t_Q}{\partial \bar{r}} \right) + \frac{Q r_0^2}{\kappa} \quad (4a)$$

$$\left. \begin{aligned} \frac{\partial t_Q}{\partial \bar{r}} &= 0 \text{ at } \bar{r} = 0 \text{ and } \bar{r} = 1 \\ t_Q(0, \bar{r}) &= 0 \end{aligned} \right\} \quad (4b)$$

$$\frac{\partial t_q}{\partial \zeta} = \frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} \left( \bar{r} \frac{\partial t_q}{\partial \bar{r}} \right) \quad (5a)$$

$$\left. \begin{aligned} \frac{\partial t_q}{\partial \bar{r}} &= 0 \quad \text{at } \bar{r} = 0 \\ \frac{\partial t_q}{\partial \bar{r}} &= \frac{q_w r_0}{\kappa} \quad \text{at } \bar{r} = 1 \\ t_q(0, \bar{r}) &= t_e \end{aligned} \right\} \quad (5b)$$

These two problems (defined by eqs. (4) and (5)) will be treated separately, and the results combined to yield information for the general situation.

Uniform internal heat generation with tube insulated. - Equation (4a) is a nonhomogeneous second-order partial differential equation. By superposition, the solution is expressed as the sum of a particular solution  $t_{Q,p}$  and a complementary solution  $t_{Q,c}$ ; that is,  $t_Q = t_{Q,p} + t_{Q,c}$  where  $t_{Q,p}$  satisfies

$$\frac{\partial t_{Q,p}}{\partial \zeta} = \frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} \left( \bar{r} \frac{\partial t_{Q,p}}{\partial \bar{r}} \right) + \frac{Q r_0^2}{\kappa} \quad (6a)$$

and  $t_{Q,c}$  satisfies

$$\frac{\partial t_{Q,c}}{\partial \zeta} = \frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} \left( \bar{r} \frac{\partial t_{Q,c}}{\partial \bar{r}} \right) \quad (6b)$$

The particular solution of equation (6a) may be obtained by inspection:

$$t_{Q,p} = \frac{Q r_0^2}{\kappa} \zeta \quad (7)$$

This form of solution identically satisfies the boundary conditions (eq. (4b)). The solution for  $t_{Q,c}$  is found by using a product solution in conjunction with a Fourier series expansion of the entrance condition, which is  $t_{Q,c}(0, \bar{r}) = 0$ . The final solution for  $t_{Q,c}$  is



$$t_{Q,c}(\zeta, \bar{r}) = 0 \quad - \quad (8)$$

Now that  $t_{Q,p}$  and  $t_{Q,c}$  are known, they can be superposed to obtain the solution for  $t_Q$  that applies over the entire length of the tube, which is

$$\frac{t_Q}{\frac{Qr_0^2}{\kappa}} = \zeta \quad (9)$$

The local bulk fluid temperature  $t_{Q,b}(\zeta)$  along the tube length, for a uniform heat source, is given by

$$\rho U c_p [t_{Q,b}(\zeta) - t_{Q,b}(0)] = Qx$$

or, since  $t_{Q,b}(0, \bar{r}) = 0$ ,

$$t_{Q,b}(\zeta) = \frac{Q}{\rho U c_p} x = \frac{Qr_0^2}{\kappa} \zeta \quad (10)$$

Combining equations (9) and (10) yields

$$\frac{t_Q - t_{Q,b}}{\frac{Qr_0^2}{\kappa}} = 0 \quad (11)$$

That is, the difference between the local fluid and bulk temperatures is zero for all  $\zeta$  and all  $\bar{r}$ .

Uniform wall heat transfer without internal heat generation. - Next is considered the situation where there is heat transfer  $q_w$  at the tube wall but no internal heat generation. The temperature  $t_q(x, r)$  is the solution to equations (5). Equation (5a) is of exactly the same form as the heat-conduction equation for the unsteady temperature distribution within an infinite cylinder with a constant heat flux at the surface (ref. 27, p. 203). The axial coordinate  $x$  is analogous to time in the transient heat-conduction problem. In reference 27, the solution is given for the transient temperature distribution in an infinite cylinder when the temperature is initially zero before a step in wall heat flux is introduced. This result immediately yields the solution that applies in both the thermal entrance and fully developed regions for slug flow in a circular tube where the temperature

of the fluid entering the heated section of the tube (commencing at  $x = 0$ ) is  $t_e$ :

$$\frac{t_q(\xi, \bar{r}) - t_e}{\frac{q_w r_0}{\kappa}} = 2\xi + \frac{1}{2} \bar{r}^2 - \frac{1}{4} + \sum_{j=1}^{\infty} A_j R_j e^{-\beta_j^2 \xi} \quad (12)$$

where the coefficients  $A_j$  and eigenfunctions  $R_j$  are given by

$$A_j = \frac{2}{\beta_j^2 J_0(\beta_j)} \quad (13)$$

$$R_j = J_0(\beta_j, \bar{r}) \quad (14)$$

and where  $J_0$  is the Bessel function of order zero of the first kind. The eigenvalues  $\beta_j$  ( $j = 1, 2, \dots$ ) are the positive roots of

$$J_1(\beta_j) = 0 \quad (15)$$

where  $J_1$  is the Bessel function of order one of the first kind. The first five roots of equation (15) are listed, for example, in reference 27. It will be useful to define a new coefficient  $\bar{A}_j$ , which is defined as the product of  $A_j$  and  $R_j(1, \beta_j) = J_0(\beta_j)$ :

$$\bar{A}_j = \frac{2}{\beta_j^2} \quad (16)$$

Table I(a) shows the first five eigenvalues  $\beta_j$  and coefficients  $\bar{A}_j$  for uniform flow in a round tube with wall heat flux.

Internal heat generation with uniform wall heat transfer. - The complete solution for the temperature  $t(\xi, \bar{r})$  that applies in the presence of internal heat generation and uniform wall heat flux is found by combining the solutions for  $t_Q$  and  $t_q$  in accordance with equation (3) to give

$$t(\xi, \bar{r}) - t_e = \left( \frac{Q r_0^2}{\kappa} + \frac{2 q_w r_0}{\kappa} \right) \xi + \frac{q_w r_0}{\kappa} \left( \frac{1}{2} \bar{r}^2 - \frac{1}{4} \right) + \frac{q_w r_0}{\kappa} \sum_{j=1}^{\infty} \bar{A}_j R_j e^{-\beta_j^2 \xi} \quad (17)$$

TABLE I. - EIGENVALUES AND COEFFICIENTS  
FOR UNIFORM FLOW IN A CIRCULAR TUBE

(a) Uniform wall heat flux and no internal heat generation.		(b) Uniform internal heat generation and zero wall temperature.	
Eigenvalue		Eigenvalue	
$\beta_1$	3.832	$\gamma_1$	2.405
$\beta_2$	7.016	$\gamma_2$	5.520
$\beta_3$	10.17	$\gamma_3$	8.654
$\beta_4$	13.32	$\gamma_4$	11.79
$\beta_5$	16.47	$\gamma_5$	14.93
Coefficient		Coefficient	
$\bar{A}_1$	-0.13620	$\bar{b}_1$	0.34578
$\bar{A}_2$	-.04063	$\bar{b}_2$	.06564
$\bar{A}_3$	-.01934	$\bar{b}_3$	.02671
$\bar{A}_4$	-.01127	$\bar{b}_4$	.01439
$\bar{A}_5$	-.00737	$\bar{b}_5$	.00897

Equation (17) may be written in dimensionless form as

$$\frac{t(\zeta, \bar{r}) - t_e}{\frac{q_w r_0}{K}} = 2(1 + M)\zeta + \frac{1}{2}\bar{r}^2 - \frac{1}{4} + \sum_{j=1}^{\infty} \bar{A}_j \bar{R}_j e^{-\beta_j^2 \zeta} \quad (18)$$

where

$$M = \frac{Q r_0}{2 q_w} \quad (19)$$

The parameter  $M$  is the ratio of internal heat evolution to the heat transferred at the tube wall and gives a measure of the relative importance, in connection with temperature development, of internal heat generation in the presence of wall heat transfer.

When the wall heat flux is specified, the longitudinal variation of wall temperature is the unknown quantity that is of practical design interest. The wall temperature variation is found by evaluating equation (18) at  $\bar{r} = 1$  with the result

$$\frac{t_w(\zeta) - t_e}{\frac{q_w r_0}{K}} = 2(1 + M)\zeta + \frac{1}{4} + \sum \bar{A}_j e^{-\beta_j^2 \zeta} \quad (20)$$

Another useful form of equation (20) is obtained by introducing the fluid bulk temperature  $t_b(\zeta)$ , which is given by

$$t_b(\zeta) - t_e = t_{Q,b} + (t_{q,b} - t_e)$$

The bulk temperature  $t_{Q,b}$  is given by equation (10), while for uniform wall heat flux

$$t_{q,b} = t_e + \frac{2q_w}{\rho U c_p r_0} x = t_e + 2 \frac{q_w r_0}{\kappa} \zeta$$

The solution for  $t_b(\zeta)$  is then given as

$$t_b(\zeta) - t_e = \left( \frac{Q r_0^2}{\kappa} + \frac{2q_w r_0}{\kappa} \right) \zeta \quad (21a)$$

Dividing equation (21a) by  $q_w r_0 / \kappa$  yields

$$\frac{t_b(\zeta) - t_e}{\frac{q_w r_0}{\kappa}} = 2(1 + M)\zeta \quad (21b)$$

Then, combining equations (20) and (21b) gives

$$\frac{t_w - t_b}{\frac{q_w r_0}{\kappa}} = \frac{1}{4} + \sum_{j=1}^{\infty} \bar{A}_j e^{-\beta_j^2 \zeta} \quad (22)$$

A noteworthy feature of equation (22) is that the variation of the wall-to-bulk temperature difference along the tube length is independent of internal heat generation or, in other words, the parameter  $M$ . For the fully developed situation ( $\zeta \rightarrow \infty$ ),

$$(t_w - t_b)_d = \frac{1}{4} \frac{q_w r_0}{\kappa} \quad (23)$$

Dividing equation (22) by (23) yields the important ratio

$$\frac{t_w - t_b}{(t_w - t_b)_d} = 1 + 4 \sum_{j=1}^{\infty} \bar{A}_j e^{-\beta_j^2 \zeta} \quad (24)$$

The variation of the dimensionless bulk-to-entrance temperature difference along the tube length has been evaluated from equation (21b) for several values of the parameter  $M$  and plots are given in figure 2(a).

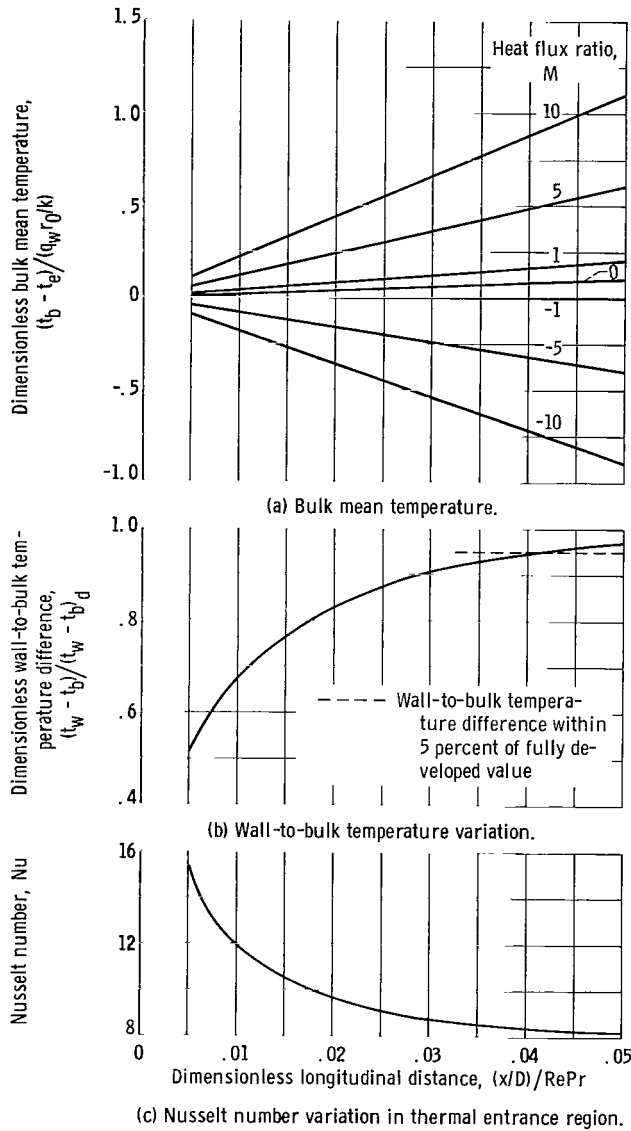


Figure 2. - Uniform internal heat generation in a circular tube with wall heat transfer.

proach within 5 percent of the fully developed value. From figure 2(b), the value of  $(x/D)/\text{RePr}$  corresponding to an ordinate of 0.95 is approximately 0.042 and, as noted previously, is independent of the parameter  $M$ .

It is customary to represent heat-transfer results in terms of a heat-transfer coefficient  $h \equiv q_w/(t_w - t_b)$  and a Nusselt number  $Nu \equiv hD/\kappa$ . When equation (22) is used,

Positive and negative values of the parameter  $M = Qr_0/2q_w$  are considered in figure 2(a). It is supposed that  $Q$  is positive (a heat source). A positive value of  $M$ , therefore, implies that  $q_w$  is positive; that is, the heat is being transferred from the wall to the fluid. A negative value of  $M$ , on the other hand, implies that  $q_w$  is negative or that heat is being transferred from the fluid to the wall. For positive  $M$ , consequently, internal heat generation and wall heat transfer reinforce one another to produce values of  $t_b - t_e$  that are larger than those obtained in the absence of internal heat generation. Conversely, for negative  $M$ , wall heat transfer opposes internal heat generation in the bulk temperature development.

For very small values of  $|M|$ , wall heat transfer dominates the temperature development, while for large values of  $|M|$ , the effects of internal heat generation dominate. This accounts for the variety of trends that are evident in figure 2(a).

Equation (24) has been evaluated by using the numerical data of table I(a) and a plot given in figure 2(b). A thermal entrance length can be defined as the heated length required for  $t_w - t_b$  to ap-

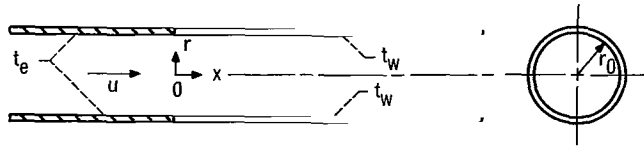


Figure 3. - Physical model and coordinate system for circular tube with uniform wall temperature.

$$Nu = \frac{8}{1 + 4 \sum_{j=1}^{\infty} \bar{A}_j e^{-\beta_j^2 \zeta}} \quad (25)$$

The Nusselt number given by equation (25) is plotted in figure 2(c). The limiting Nusselt number value is 8, which is in agreement with the tabulated asymptotic Nusselt number for slug flow in a circular tube with uniform wall heat flux (ref. 28).

The fact that the wall-to-bulk temperature and the Nusselt modulus variations are independent of the internal heat generation is due to the assumption that the velocity of the liquid metal is uniform over the pipe cross section. Since this assumption is a good approximation of the real problem of turbulent liquid-metal flow, the thermal entrance length and Nusselt modulus variation in the actual situation would be only slightly affected by the presence of internal heat sources.

### Uniform Heat Generation and Uniform Wall Temperature

Consideration is now given to the problem in which a liquid metal with a uniform temperature  $t_e$  enters a tube whose wall is maintained at a uniform temperature  $t_w$  different from the entrance value (fig. 3). In addition, a uniform heat generation begins at  $x = 0$  within the fluid. The velocity profile is again approximated by equation (1) and the energy transfer processes in the fluid are still governed by equation (2). The linearity of the energy equation enables the fluid temperature  $t(x, r)$  to again be written as the sum of two parts:

$$t(x, r) = t_Q(x, r) + t_T(x, r) \quad (26a)$$

where  $t_Q$  corresponds to the problem in which a heat-generating fluid at  $t = 0$  enters a tube whose wall temperature is also  $t = 0$ , and  $t_T$  corresponds to the problem in which a nongenerating fluid at  $t_e$  enters a tube whose wall temperature is  $t_w$ . It is convenient to rewrite equation (26a) in the dimensionless form

$$\frac{t(x, r) - t_w}{t_e - t_w} = \frac{t_Q(x, r)}{t_e - t_w} + \frac{t_T(x, r) - t_w}{t_e - t_w} \equiv T_Q + T_T \quad (26b)$$

The dimensionless governing equations and boundary conditions for  $T_Q$  and  $T_T$  are

$$\frac{\partial T_Q}{\partial \zeta} = \frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} \left( \bar{r} \frac{\partial T_Q}{\partial \bar{r}} \right) + \frac{Qr_0^2}{\kappa(t_e - t_w)} \quad (27a)$$

$$\left. \begin{aligned} \frac{\partial T_Q}{\partial \bar{r}} &= 0 \quad \text{at } \bar{r} = 0 \\ T_Q &= 0 \quad \text{at } \bar{r} = 1 \\ T_Q &= 0 \quad \text{at } \zeta = 0 \end{aligned} \right\} \quad (27b)$$

$$\frac{\partial T_T}{\partial \zeta} = \frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} \left( \bar{r} \frac{\partial T_T}{\partial \bar{r}} \right) \quad (28a)$$

$$\left. \begin{aligned} \frac{\partial T_T}{\partial \bar{r}} &= 0 \quad \text{at } \bar{r} = 0 \\ T_T &= 0 \quad \text{at } \bar{r} = 1 \\ T_T &= 1 \quad \text{at } \zeta = 0 \end{aligned} \right\} \quad (28b)$$

The solution proceeds in the same manner as for the uniform wall heat flux case; that is, the two problems are treated separately and the results combined to yield information for the general situation.

Uniform internal heat generation with a wall temperature  $t_w$  of zero. - Equation (27a) is of exactly the same form as the heat-conduction equation for the unsteady temperature distribution within an infinite circular cylinder with heat generated within it (ref. 27, p. 204). Again the axial coordinate  $x$  is analogous to time in the transient heat-conduction problem. In reference 27, the solution is given for the transient temperature distribution in an infinite cylinder when the initial and surface temperatures are zero and a constant heat generation per unit time per unit volume is suddenly imposed. This result immediately yields the solution for uniform flow in a circular tube where the entering fluid and wall temperatures are zero and an internal heat generation commences at  $x = 0$ . The solution applying over the entire length of the tube is obtained as

$$\frac{T_Q(\zeta, \bar{r})}{\frac{Qr_0^2}{\kappa(t_e - t_w)}} = \frac{1}{4} (1 - \bar{r}^2) + \sum_{k=1}^{\infty} b_k \tilde{R}_k e^{-\gamma_k^2 \zeta} \quad (29)$$

where the series coefficients  $b_k$  and eigenfunctions  $\tilde{R}_k$  are given by, respectively,

$$b_k = - \frac{2}{\gamma_k^3 J_1(\gamma_k)} \quad (30)$$

$$\tilde{R}_k = J_0(\gamma_k \bar{r}) \quad (31)$$

The eigenvalues  $\gamma_k$  ( $k = 1, 2, \dots$ ) are obtained as the positive roots of

$$J_0(\gamma_k) = 0 \quad (32)$$

The first five roots of equation (32) are also listed in reference 27. A new coefficient  $\bar{b}_k$  can conveniently be defined as the product of  $b_k$  and  $\left(\frac{d\tilde{R}}{d\bar{r}}\right)_{\bar{r}=1, \gamma=\gamma_k}$

$$\bar{b}_k = \frac{2}{\gamma_k^2} \quad (33)$$

Table I(b) (p. 9) shows the first five eigenvalues  $\gamma_k$  and constants  $\bar{b}_k$ . Subsequent analyses will require a knowledge of the fluid mixed-mean temperature  $t_{Q,b}$ , which may be found from the definition

$$T_{Q,b}(\zeta) \equiv \frac{t_{Q,b}}{t_e - t_w} = \frac{\int_0^1 u T_Q(\zeta, \bar{r}) \bar{r} d\bar{r}}{\int_0^1 u \bar{r} d\bar{r}} \quad (34)$$

However,  $u = U = \text{constant}$  and  $\int_0^1 \bar{r} d\bar{r} = 1/2$ , so that



$$T_{Q,b}(\zeta) = 2 \int_0^1 T_Q(\zeta, \bar{r}) \bar{r} d\bar{r} \quad (35)$$

Introducing equation (29) into equation (35) yields

$$T_{Q,b} = \frac{Qr_0^2}{\kappa(t_e - t_w)} \left( \frac{1}{8} + 2 \sum_{k=1}^{\infty} b_k e^{-\gamma_k^2 \zeta} \int_0^1 \tilde{R}_k \bar{r} d\bar{r} \right) \quad (36)$$

The integration is carried out with the result

$$\int_0^1 \tilde{R}_k \bar{r} d\bar{r} = \frac{1}{\gamma_k} J_1(\gamma_k)$$

With this, equation (36) becomes

$$T_{Q,b} = \frac{Qr_0^2}{\kappa(t_e - t_w)} \left( \frac{1}{8} - 2 \sum_{k=1}^{\infty} \frac{\bar{b}_k}{\gamma_k^2} e^{-\gamma_k^2 \zeta} \right) = \frac{Qr_0^2}{\kappa(t_e - t_w)} \left( \frac{1}{8} - 4 \sum_{k=1}^{\infty} \frac{e^{-\gamma_k^2 \zeta}}{\gamma_k^4} \right) \quad (37)$$

The problem of a liquid metal containing no heat sources and flowing in a tube with a uniform wall temperature is considered next.

Uniform wall temperature without internal heat generation. - The temperature  $T_T(x, r)$  is the solution to equations (28), and it can be obtained directly from the solution of the corresponding heat-conduction problem (ref. 27, p. 199), wherein an infinite cylinder has a zero initial temperature and its surface temperature is step changed to a new constant value, by once again treating the axial coordinate  $x$  as analogous to time in the heat-conduction problem. The solution to equations (28) can therefore be written as

$$T_T(\zeta, \bar{r}) = \sum_{k=1}^{\infty} B_k \tilde{R}_k e^{-\gamma_k^2 \zeta} \quad (38)$$

where  $\gamma_k$  and  $\tilde{R}_k$  are the same eigenvalues and eigenfunctions obtained previously (eqs. (31) and (32)). The coefficients  $B_k$  are determined readily as

$$B_k = \frac{2}{\gamma_k J_1(\gamma_k)} \quad (39)$$

A new coefficient  $\bar{B}_k$ , which is defined as the product of  $B_k$  and  $\left(\frac{d\tilde{R}}{d\bar{r}}\right)_{\bar{r}=1, \gamma=\gamma_k}$ , is introduced for convenience as

$$\bar{B}_k \equiv B_k \left(\frac{d\tilde{R}}{d\bar{r}}\right)_{\bar{r}=1, \gamma=\gamma_k} = -2 \quad (40)$$

The mixed-mean temperature  $t_{T, b}$  of the fluid is given by its definition as

$$\frac{t_{T, b} - t_w}{t_e - t_w} \equiv T_{T, b} = \frac{\int_0^1 U T_T \bar{r} d\bar{r}}{\int_0^1 U \bar{r} d\bar{r}} = 2 \int_0^1 T_T \bar{r} d\bar{r} \quad (41)$$

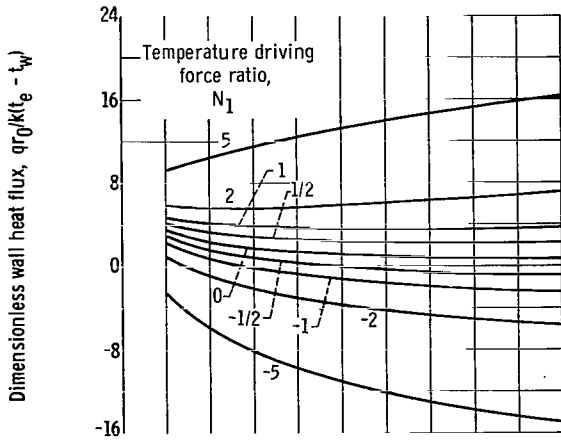
Introducing equation (38) for  $T_T(\zeta, \bar{r})$  into equation (41) and carrying out the integration yield

$$T_{T, b} = -2 \sum_{k=1}^{\infty} \frac{\bar{B}_k}{\gamma_k^2} e^{-\gamma_k^2 \zeta} = 4 \sum_{k=1}^{\infty} \frac{1}{\gamma_k^2} e^{-\gamma_k^2 \zeta} \quad (42)$$

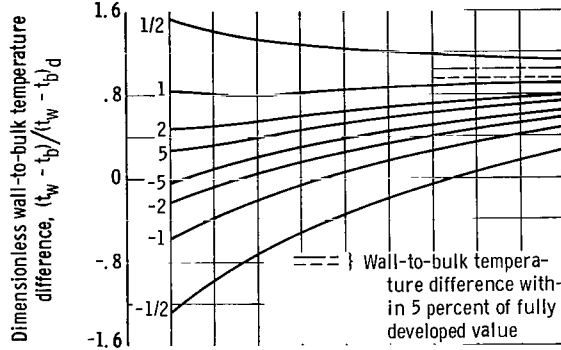
Internal heat generation with uniform wall temperature. - The complete solution for the temperature  $t(\zeta, \bar{r})$  that applies for a heat-generating fluid flowing in a pipe with uniform wall temperature  $t_w$  is found by combining the solutions for  $T_Q$  and  $T_T$  in accordance with equation (26b) to yield

$$\frac{t(\zeta, \bar{r}) - t_w}{t_e - t_w} \equiv T(\zeta, \bar{r}) = \frac{Qr_0^2}{\kappa(t_e - t_w)} \left[ \frac{1}{4} (1 - \bar{r}^2) + \sum_{k=1}^{\infty} b_k \tilde{R}_k e^{-\gamma_k^2 \zeta} \right] + \sum_{k=1}^{\infty} B_k \tilde{R}_k e^{-\gamma_k^2 \zeta} \quad (43)$$

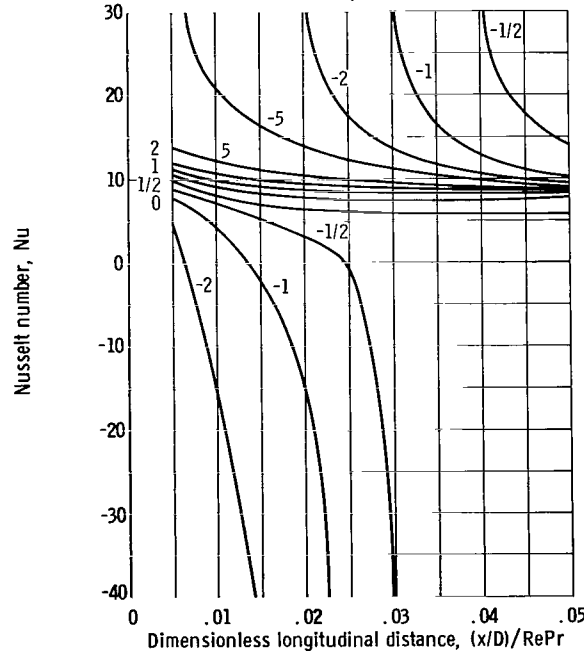
When the tube wall temperature is specified, the longitudinal variation of the wall



(a) Wall heat flux variation.



(b) Wall-to-bulk temperature variation.



(c) Nusselt number variation.

Figure 4. - Flow in a circular tube with uniform internal heat generation and uniform wall temperature.

heat flux from liquid to wall required to maintain the wall temperature constant is of practical design interest. The heat-transfer rate at the tube wall is given by

$$q = \left( -\kappa \frac{\partial t}{\partial r} \right)_{r=r_0} = \left[ -\frac{\kappa(t_e - t_w)}{r_0} \frac{\partial T}{\partial \bar{r}} \right]_{\bar{r}=1} \quad (44)$$

The heat-transfer rate  $q$  can be found by differentiating equation (43) with respect to  $\bar{r}$  and evaluating the result at  $\bar{r} = 1$  in accordance with equation (44) to obtain

$$\begin{aligned} \frac{qr_0}{\kappa(t_e - t_w)} &= \frac{Qr_0^2}{\kappa(t_e - t_w)} \left( \frac{1}{2} - \sum_{k=1}^{\infty} \bar{b}_k e^{-\gamma_k^2 \zeta} \right) \\ &+ 2 \sum_{k=1}^{\infty} e^{-\gamma_k^2 \zeta} = N_1 \left( 4 - 8 \sum_{k=1}^{\infty} \bar{b}_k e^{-\gamma_k^2 \zeta} \right) \\ &+ 2 \sum_{k=1}^{\infty} e^{-\gamma_k^2 \zeta} \end{aligned} \quad (45)$$

where  $N_1 \equiv Qr_0^2 / 8\kappa(t_e - t_w)$ . The variation of the dimensionless wall heat-transfer rate along the length of the tube has been evaluated from equation (45) and the numerical data listed in table I(b) (p. 9); plots are given in figure 4(a) for several values of the parameter  $N_1$ .

The parameter  $N_1 = Qr_0^2 / 8\kappa(t_e - t_w)$  is the ratio of internal heat generation to the temperature driving force  $t_e - t_w$ . It is again supposed that  $Q$  is positive, but clearly  $t_e - t_w$  may be either positive

or negative. Both positive and negative values of the parameter  $N_1$ , therefore, are considered in the figure.

Inspection of the figure reveals a variety of trends for the wall heat flux variation. For  $N_1 = 0$ , which implies no internal heat generation, the wall heat flux  $q$  decreases with  $x$ . For  $N_1 \neq 0$ ,  $q$  may decrease with  $x$ , increase with  $x$ , or at first decrease and then increase with  $x$ , depending on the magnitude and sign of  $N_1$ . When  $N_1$  is positive, the heat flux is increased, at a given longitudinal position  $x$ , over that required when  $N_1 = 0$ . For small positive values of  $N_1$ , the heat flux is essentially influenced by the temperature driving force  $t_e - t_w$  for small  $x$ , while for large  $x$  the heat flux is essentially determined by the internal heat generation. For a large positive value of  $N_1$ , the internal heat generation dominates over the range shown for  $x$ .

For negative values of  $N_1$ ,  $q$  may change sign at some position along the tube length, which indicates a change in direction of the heat transfer at that location along the wall. For a large negative value of  $N_1$ , the wall heat transfer is essentially determined by the internal heat generation and, for large  $x$ , approaches in magnitude the local heat transfer required for the corresponding positive value of  $N_1$ .

The fluid mixed-mean temperature in the present situation is given by

$$\frac{t_b(\zeta) - t_w}{t_e - t_w} = T_{Q,b} + T_{T,b} = \frac{Qr_0^2}{\kappa(t_e - t_w)} \left( \frac{1}{8} - 4 \sum_{k=1}^{\infty} \frac{1}{\gamma_k^4} e^{-\gamma_k^2 \zeta} \right) + 4 \sum_{k=1}^{\infty} \frac{1}{\gamma_k^2} e^{-\gamma_k^2 \zeta} \quad (46)$$

and for the fully developed situation,

$$\left( \frac{t_b - t_w}{t_e - t_w} \right)_d = \frac{1}{8} \frac{Qr_0^2}{\kappa(t_e - t_w)} = N_1 \quad (47)$$

Dividing equation (46) by equation (47) yields the ratio

$$\frac{t_b - t_w}{(t_b - t_w)_d} = 1 - 32 \sum_{k=1}^{\infty} \frac{1}{\gamma_k^4} e^{-\gamma_k^2 \zeta} + \frac{4}{N_1} \sum_{k=1}^{\infty} \frac{1}{\gamma_k^2} e^{-\gamma_k^2 \zeta} \quad (48)$$

This relation has been evaluated by using the numerical data of table I(b) and a plot given in figure 4(b) for representative values of the parameter  $N_1$ . A horizontal line delineates

ting the condition  $(t_b - t_w)/(t_b - t_w)_d = 0.95$  or  $1.05$  (depending on whether the wall-to-mixed-mean temperature ratio approaches unity from above or below) has been drawn in the figure to aid in defining the thermal entrance length.

From the figure it is seen that the wall-to-mixed-mean temperature ratio depends on the parameter  $N_1$  which in turn, for a given value of  $t_e - t_w$ , depends on the magnitude of the internal heat generation. This result differs from that obtained in the case of uniform wall heat transfer, where the wall-to-bulk temperature development was independent of internal heat generation.

The thermal entrance length increases as  $N_1$  takes on progressively larger positive values. For negative values of  $N_1$ , the thermal entrance length decreases as  $N_1$  assumes larger magnitudes. For moderate negative values of  $N_1$ , the wall-to-mixed-mean temperature difference varies substantially along the length of the tube. The temperature difference becomes zero when the mixed-mean temperature of the fluid is equal to the wall temperature. Since the Nusselt number is inversely proportional to the wall-to-mixed-mean temperature difference, infinities in the Nusselt number will occur when the differences are zero.

Finally, it should be noted that the wall-to-mixed-mean temperature development is not shown for  $N_1 = 0$ . The ratio is not appropriate for  $N_1 = 0$  since for this case the fully developed value is zero (eq. (47)).

The Nusselt number  $Nu$  may be written

$$Nu = \frac{hD}{\kappa} = \frac{q}{t_b - t_w} \frac{2r_0}{\kappa} = 2 \frac{qr_0}{\kappa(t_e - t_w)} \frac{t_e - t_w}{t_b - t_w} \quad (49)$$

With the use of equations (45) and (46), the Nusselt number may be expressed as

$$Nu = 2 \frac{4 - 8 \sum_{k=1}^{\infty} \frac{\bar{b}_k}{\gamma_k^2} e^{-\gamma_k^2 \zeta} + \frac{2}{N_1} \sum_{k=1}^{\infty} e^{-\gamma_k^2 \zeta}}{1 - 16 \sum_{k=1}^{\infty} \frac{\bar{b}_k}{\gamma_k^2} e^{-\gamma_k^2 \zeta} + \frac{4}{N_1} \sum_{k=1}^{\infty} \frac{1}{\gamma_k^2} e^{-\gamma_k^2 \zeta}} \quad (50)$$

The numerical results for  $Nu$  are plotted in figure 4(c) with  $N_1$  appearing as the family parameter. An overall inspection of the figure reveals that for positive values of  $N_1$  the local Nusselt numbers, and hence the local heat transfer coefficients, are large in the neighborhood of the tube entrance and decrease continuously with increasing downstream distance. When  $N_1$  is negative, the Nusselt number becomes infinite at the

point where the fluid mixed-mean temperature equals the wall temperature and becomes negative when the mixed-mean temperature is greater than the wall temperature. An appreciable variation of the heat-transfer coefficient along the tube length therefore occurs for moderate negative values of  $N_1$ . This variation exhibited in the figure makes the utility of a heat-transfer coefficient under the conditions of a specified wall temperature and internal heat generation questionable. Independent consideration of equations (45) and (46) may prove more useful for a quantitative heat-transfer evaluation in this situation.

It is of interest, however, to examine the fully developed heat-transfer conditions. For nonzero values of  $N_1$ , the following is obtained for the fully developed Nusselt number  $Nu_d(\zeta \rightarrow \infty)$ ;

$$Nu_d = 8 \quad (51a)$$

It is noted that this value is independent of the magnitude of  $N_1$ .

When  $N_1 = 0$ , the Nusselt number at any longitudinal location is obtained from equation (50) as

$$Nu = \frac{\sum_{k=1}^{\infty} e^{-\gamma_k^2 \zeta}}{\sum_{k=1}^{\infty} \frac{1}{\gamma_k^2} e^{-\gamma_k^2 \zeta}}$$

The fully developed heat-transfer condition occurs for  $\zeta$  sufficiently large so that only the first term of the series need be considered. The fully developed Nusselt number then follows as

$$Nu_d = \gamma_1^2 = 5.78 \quad (51b)$$

which agrees with tabulated values (ref. 28).

## HEAT TRANSFER TO A LIQUID METAL FLOWING BETWEEN PARALLEL PLATES

This section of the investigation will be subdivided into two subsections dealing with (1) uniform internal heat generation and uniform, but unequal, wall heat fluxes and

(2) uniform heat generation and uniform, but unequal, plate temperatures.

Asymmetric heating or cooling is often encountered when dealing with forced convection heat transfer for flow through flat rectangular ducts; for example, when the temperature of the environment at one side differs from that at the other side of the channel, unequal wall heat fluxes exist. Another example of an instance when this situation may arise is when heat leakage or addition through insulation occurs.

A uniform wall temperature boundary condition is encountered, for example, when a coolant undergoing a change of phase is employed in an adjacent flow passage. The temperature of the walls is therefore maintained essentially constant. Different coolants undergoing a change of phase, moreover, may be so employed in adjacent flow passages that the temperature of one plate of the channel differs from that of the other plate.

### Uniform Heat Generation and Uniform Unequal Wall Heat Transfer

The geometry considered is shown in figure 5. In the section of the channel to the right of  $x = 0$ , a heating process takes place that includes heat generation in the fluid and unsymmetrical heating at the duct walls.

The established turbulent velocity profile is still assumed approximated by equation (1), while the differential equation describing convective heat transfer between the plates is

$$U \frac{\partial t}{\partial x} = \frac{\kappa}{\rho c_p} \frac{\partial^2 t}{\partial y^2} + \frac{Q}{\rho c_p} \quad (52)$$

Because of the linearity of equation (52), the temperature distribution  $t(x, y)$  is written as in the circular tube problem as the sum of two parts:

$$t(x, y) = t_Q(x, y) + t_q(x, y) \quad (53)$$

The temperature  $t_Q(x, y)$  is the temperature of a heat-generating fluid flowing between in-

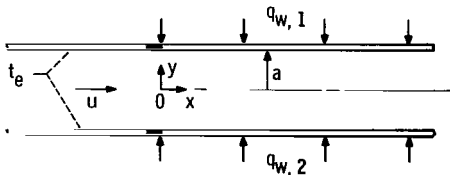


Figure 5. - Physical model and coordinate system for parallel-plate channel with unsymmetrical wall heat transfer.

ulated plates ( $q_{w,1} = q_{w,2} = 0$ ), while  $t_q$  corresponds to the situation where there are unsymmetrical heat-transfer rates  $q_{w,1}$  and  $q_{w,2}$  at the plates but no internal heat generation ( $Q = 0$ ).

The dimensionless equations and boundary conditions used to determine  $t_Q(x, y)$  and  $t_q(x, y)$  are, respectively,

$$\frac{\partial t_Q}{\partial \xi} = \frac{\partial^2 t_Q}{\partial \bar{y}^2} + \frac{Qa^2}{\kappa} \quad (54a)$$

$$\left. \begin{aligned} \frac{\partial t_Q}{\partial \bar{y}} &= 0 \text{ at } \bar{y} = 0 \text{ and } \bar{y} = 1 \\ t_Q(0, \bar{y}) &= 0 \end{aligned} \right\} \quad (54b)$$

$$\frac{\partial t_q}{\partial \xi} = \frac{\partial^2 t_q}{\partial \bar{y}^2} \quad (55a)$$

$$\left. \begin{aligned} \frac{\partial t_q}{\partial \bar{y}} &= -\frac{q_{w, 2} a}{\kappa} \text{ at } \bar{y} = -1 \\ \frac{\partial t_q}{\partial \bar{y}} &= \frac{q_{w, 1} a}{\kappa} \text{ at } \bar{y} = 1 \\ t_q(0, \bar{y}) &= t_e \end{aligned} \right\} \quad (55b)$$

The method of solution for the general situation is identical to that employed for the circular tube problem. That is, the two problems are studied independently and the results combined in accordance with equation (53) since, if the individual temperature fields satisfy the linear energy equation, their sum does also.

Uniform internal heat generation with plates insulated. - Equation (54a) is a non-homogeneous second-order partial differential equation. By superposition, the solution is expressed as the sum of a particular solution  $t_{Q, p}$  and a complementary solution  $t_{Q, c}$ . The particular solution satisfies the equation

$$\frac{\partial t_{Q, p}}{\partial \xi} = \frac{\partial^2 t_{Q, p}}{\partial \bar{y}^2} + \frac{Qa^2}{\kappa} \quad (56a)$$

and  $t_{Q, c}$  satisfies the equation



$$\frac{\partial t_{Q,c}}{\partial \xi} = \frac{\partial^2 t_{Q,c}}{\partial \bar{y}^2} \quad (56b)$$

The particular solution of equation (56a) that satisfies the boundary conditions (eq. (54b)) is obtained by inspection as

$$t_{Q,p} = \frac{Qa^2}{\kappa} \xi \quad (57)$$

The solution for  $t_{Q,c}$  is found by using a product solution in conjunction with a Fourier series expansion of the entrance condition ( $t_{Q,c}(0, \bar{y}) = 0$ ). The solution for  $t_{Q,c}$  is

$$t_{Q,c} = 0 \quad (58)$$

The final expression for the temperature distribution at any location in the channel is obtained by summing  $t_{Q,p}$  and  $t_{Q,c}$  to give

$$t_Q(\xi, \bar{y}) = \frac{Qa^2}{\kappa} \xi \quad (59)$$

The local bulk temperature along the channel length is given by

$$\rho U c_p [t_{Q,b}(\xi) - t_{Q,b}(0)] = Qx$$

or, alternatively,

$$t_{Q,b}(\xi) = \frac{Q}{\rho U c_p} x = \frac{Qa^2}{\kappa} \xi \quad (60)$$

Unsymmetrical wall heat transfer without internal heat generation. - The temperature  $t_q(\xi, \bar{y})$  is the solution to equations (55a) and (55b). To obtain a solution for  $t_q$  that will apply over the entire length of the channel, it is convenient to break  $t_q$  into two parts. The first part is the thermally fully developed solution  $t_{q,d}$ . The second part is the thermal entrance region solution  $t_q^*$  that is added to  $t_{q,d}$  in order to obtain temperatures in the region near the entrance of the channel. The temperature distribution at any location in the channel is then given by the sum

$$t_q(x, y) = t_{q,d}(x, y) + t_q^*(x, y)$$

The fully developed heat-transfer condition is given by

$$\frac{\partial t_{q,d}}{\partial x} = \frac{q_{w,1} + q_{w,2}}{2\rho U c_p a} \quad (61)$$

This equation may be integrated and expressed in the dimensionless form

$$\frac{t_{q,d} - t_e}{\frac{q_{w,1}(1 + \epsilon)a}{2\kappa}} = \xi + G(\bar{y}) \quad (62)$$

where  $\epsilon \equiv q_{w,2}/q_{w,1}$ . When equation (62) is substituted into equation (55a) written for  $t_{q,d}$ , the function  $G(\bar{y})$  is readily found to satisfy the differential equation

$$\frac{d^2 G}{d\bar{y}^2} = 1 \quad (63a)$$

subject to the boundary conditions

$$\frac{dG}{d\bar{y}} = -\frac{2\epsilon}{1 + \epsilon} \quad \text{at } \bar{y} = -1 \quad (63b)$$

$$\frac{dG}{d\bar{y}} = \frac{2}{1 + \epsilon} \quad \text{at } \bar{y} = 1 \quad (63c)$$

From an overall energy balance on the fluid for the length of channel from 0 to  $x$ , the following additional condition is obtained:

$$\frac{1}{2} \int_{-1}^1 (t_{q,d} - t_e) d\bar{y} = \frac{q_{w,1} + q_{w,2}}{2\rho U c_p a} x$$

or, alternatively,

$$\int_{-1}^1 G(\bar{y}) d\bar{y} = 0 \quad (63d)$$

The solution for the distribution function  $G(\bar{y})$  is easily obtained as

$$G(\bar{y}) = \frac{1}{2} \bar{y}^2 - \frac{1}{6} + \frac{1 - \epsilon}{1 + \epsilon} \bar{y} \quad (64)$$

so that the solution for  $t_{q,d}$  is

$$\frac{t_{q,d} - t_e}{q_{w,1}(1 + \epsilon)a} = \xi + \frac{1}{2} \bar{y}^2 - \frac{1}{6} + \frac{1 - \epsilon}{1 + \epsilon} \bar{y} \quad (65)$$

$$2\kappa$$

For the entrance region, use is made of a difference temperature  $t_q^*$ . From the linearity of the energy equation, the equation for  $t_q^*$  is the same as equation (55a):

$$\frac{\partial t_q^*}{\partial \xi} = \frac{\partial^2 t_q^*}{\partial \bar{y}^2} \quad (66a)$$

Since the wall heat addition has already been accounted for in the fully developed solution, the boundary conditions for  $t_q^*$  are that no heat is transferred at the walls:

$$\frac{\partial t_q^*}{\partial \bar{y}} = 0 \quad \text{at } \bar{y} = -1 \quad \text{and } \bar{y} = 1 \quad (66b)$$

At  $\xi = 0$  the condition is

$$t_q(0, \bar{y}) = t_e = t_{q,d}(0, \bar{y}) + t_q^*(0, \bar{y})$$

or, by rearranging,

$$t_q^*(0, \bar{y}) = -[t_{q,d}(0, \bar{y}) - t_e] = -G(\bar{y}) \quad (66c)$$

The solution of equation (66a) that will satisfy the conditions of equation (66b) can be found by using a product solution that leads to a separation of variables. This will have the form

$$\frac{t_q^*(\xi, \bar{y})}{q_{w,1}(1+\epsilon)a} = \sum_{m=1}^{\infty} C_m W_m e^{-\eta_m^2 \xi} \quad (67)$$

where  $\eta_m$  and  $W_m(\bar{y}, \eta_m)$  are the eigenvalues and eigenfunctions of the Sturm-Liouville problem:

$$\left. \begin{aligned} \frac{d^2 W_m}{d\bar{y}^2} + \eta_m^2 W_m &= 0 \\ \frac{dW_m}{d\bar{y}} &= 0 \text{ at } \bar{y} = -1 \text{ and } \bar{y} = 1 \end{aligned} \right\} \quad (68)$$

The coefficients  $C_m$  are found by applying the boundary condition at  $\xi = 0$  and by using the properties of the Sturm-Liouville system with the result

$$C_m = - \frac{\int_{-1}^1 \left( \frac{1}{2} \bar{y}^2 - \frac{1}{6} + \frac{1-\epsilon}{1+\epsilon} \bar{y} \right) W_m d\bar{y}}{\int_{-1}^1 W_m^2 d\bar{y}} = - \frac{\int_{-1}^1 \left( \frac{1}{2} \bar{y}^2 - \frac{1}{6} \right) W_m d\bar{y}}{\int_{-1}^1 W_m^2 d\bar{y}} + \frac{\frac{1-\epsilon}{1+\epsilon} \int_{-1}^1 \bar{y} W_m d\bar{y}}{\int_{-1}^1 W_m^2 d\bar{y}} \quad (69)$$

The eigenfunctions may be even or odd in  $\bar{y}$ . If  $W_m$  is an even function, the second term on the right side of equation (69) disappears, while if  $W_m$  is an odd function, the first term disappears. The constants  $C_m$  can therefore be divided into two classes  $C_{m,e}$  and  $C_{m,o}$  given by the results

$$C_{m,e} = - \frac{\int_{-1}^1 \left( \frac{1}{2} \bar{y}^2 - \frac{1}{6} \right) Y_m d\bar{y}}{\int_{-1}^1 Y_m^2 d\bar{y}} = - \frac{\int_0^1 \left( \frac{1}{2} \bar{y}^2 - \frac{1}{6} \right) Y_m d\bar{y}}{\int_0^1 Y_m^2 d\bar{y}} \quad (70)$$

$$\frac{C_{m,o}}{(1-\epsilon)} \equiv C_{m,o}^* = - \frac{\int_{-1}^1 \bar{y} Z_m d\bar{y}}{\int_{-1}^1 Z_m^2 d\bar{y}} = - \frac{\int_0^1 \bar{y} Z_m d\bar{y}}{\int_0^1 Z_m^2 d\bar{y}} \quad (71)$$

where  $Y_m(\bar{y})$  and  $Z_m(\bar{y})$  are even and odd functions, respectively; that is,  $Y_m(\bar{y}) = Y_m(-\bar{y})$  and  $Z_m(\bar{y}) = -Z_m(-\bar{y})$ . The corresponding eigenvalues are denoted by  $\eta_{m,e}$  and  $\eta_{m,o}$ . The solution for  $t_q^*$  can therefore be written as

$$\frac{t_q^*(\xi, \bar{y})}{q_{w,1}(1+\epsilon)a} = \sum_{m=1}^{\infty} C_{m,e} Y_m e^{-\eta_{m,e}^2 \xi} + \frac{1-\epsilon}{1+\epsilon} \sum_{m=1}^{\infty} C_{m,o}^* Z_m e^{-\eta_{m,o}^2 \xi} \quad (72)$$

where the functions  $Y_m$  and  $Z_m$  are, respectively, the solutions to the Sturm-Liouville problems:

$$\left. \begin{aligned} \frac{d^2 Y_m}{d\bar{y}^2} + \eta_{m,e}^2 Y_m &= 0 \\ \frac{dY_m}{d\bar{y}} &= 0 \text{ at } \bar{y} = 0 \text{ and } \bar{y} = 1 \end{aligned} \right\} \quad (73)$$

$$\left. \begin{aligned} \frac{d^2 Z_m}{d\bar{y}^2} + \eta_{m,o}^2 Z_m &= 0 \\ Z_m(0) &= 0; \frac{dZ_m}{d\bar{y}} = 0 \text{ at } \bar{y} = 1 \end{aligned} \right\} \quad (74)$$

The condition  $Z_m(0) = 0$  in equation (74) arises from the fact that  $Z_m(\bar{y}) = -Z_m(-\bar{y})$ .

The even function  $Y(\bar{y}, \eta_{m,e})$  is the solution of equation (73) and the customary normalization convention  $Y(0) = 1$ . The solution that satisfies equation (73) and the condition  $Y(0) = 1$  is

$$Y_m = \cos \eta_{m,e} \bar{y} \quad (75)$$

where

$$\eta_{m,e} = m\pi \quad m = 1, 2, \dots \quad (76)$$

The odd distribution function  $Z(\bar{y}, \eta_{m,o})$  that satisfies equation (74) is found to be

$$Z_m = \sin \eta_{m,o} \bar{y} \quad (77)$$

where

$$\eta_{m,o} = \frac{2m+1}{2} \pi \quad m = 0, 1, \dots \quad (78)$$

The series coefficients  $C_{m,e}$  given by equation (70) are obtained by substituting  $Y_m$  from equation (75) into the integrals, integrating by parts, and utilizing equation (73). The final result is

$$C_{m,e} = - \frac{2}{\eta_{m,e}^2 \cos \eta_{m,e}} = - \frac{2(-1)^m}{(m\pi)^2} \quad (79)$$

It is convenient to define a new coefficient  $\bar{C}_{m,e}$  defined as the product of  $C_{m,e}$  and  $Y_m(1)$ :

$$\bar{C}_{m,e} \equiv C_{m,e} Y_m(1) = - \frac{2}{\eta_{m,e}^2} = - \frac{2}{(m\pi)^2} \quad (80)$$

In a similar manner it can be shown that the coefficients  $C_{m,o}^*$  are given by

$$C_{m,o}^* = - \frac{2}{\eta_{m,o} \sin \eta_{m,o}} = - \frac{2(-1)^m}{\left[ \left( m + \frac{1}{2} \right) \pi \right]^2} \quad (81)$$

and therefore

$$\bar{C}_{m,o}^* \equiv C_{m,o}^* Z_m(1) = - \frac{2}{\eta_{m,o}^2} = - \frac{2}{\left[ \left( m + \frac{1}{2} \right) \pi \right]^2} \quad (82)$$

The solution for the temperature that applies in both the entrance and fully developed regions is found by summing the solutions for  $t_{q,d}$  and  $t_q^*$  to yield the result

$$\frac{t_q(\xi, \bar{y}) - t_e}{q_{w,1}(1+\epsilon)a} = \xi + \frac{1}{2}\bar{y}^2 - \frac{1}{6} + \sum_{m=1}^{\infty} C_{m,e} \cos \eta_{m,e} \bar{y} e^{-\eta_{m,e}^2 \xi} + \frac{1-\epsilon}{1+\epsilon} \left( \bar{y} + \sum_{m=0}^{\infty} C_{m,o}^* \sin \eta_{m,o} \bar{y} e^{-\eta_{m,o}^2 \xi} \right) \quad (83)$$

The fluid bulk temperature  $t_{q,b}$  in the presence of unsymmetrical wall heating is given by the energy balance

$$2\rho c_p Ua(t_{q,b}(x) - t_e) = (q_{w,1} + q_{w,2})x \quad (84a)$$

or, alternatively,

$$\frac{t_{q,b}(\xi) - t_e}{q_{w,1}(1+\epsilon)a} = \xi \quad (84b)$$

Internal heat generation with unsymmetrical wall heat fluxes. - The fluid temperature distribution  $t(\xi, y)$  due to the combined action of internal heat generation and unsymmetrical wall heat fluxes is obtained by adding the solutions for  $t_Q$  and  $t_q$  to obtain

$$t - t_e = \left[ \frac{Qa^2}{\kappa} + \frac{q_{w,1}(1+\epsilon)a}{2\kappa} \right] \xi + \frac{q_{w,1}(1+\epsilon)a}{2\kappa} \left( \frac{1}{2}\bar{y}^2 - \frac{1}{6} \right) + \frac{q_{w,1}(1+\epsilon)a}{2\kappa} \sum_{m=1}^{\infty} C_{m,e} \cos \eta_{m,e} \bar{y} e^{-\eta_{m,e}^2 \xi} + \frac{q_{w,1}(1-\epsilon)a}{2\kappa} \left( \bar{y} + \sum_{m=0}^{\infty} C_{m,o}^* \sin \eta_{m,o} \bar{y} e^{-\eta_{m,o}^2 \xi} \right) \quad (85a)$$

or in dimensionless form

$$\frac{t - t_e}{\frac{q_{w,1}(1 + \epsilon)a}{2\kappa}} = (1 + P)\xi + \frac{1}{2}\bar{y}^2 - \frac{1}{6} + \sum_{m=1}^{\infty} C_{m,e} \cos \eta_{m,e} \bar{y} e^{-\eta_{m,e}^2 \xi} + \frac{1 - \epsilon}{1 + \epsilon} \left( \bar{y} + \sum_{m=0}^{\infty} C_{m,o}^* \sin \eta_{m,o} \bar{y} e^{-\eta_{m,o}^2 \xi} \right) \quad (85b)$$

where

$$P = \frac{2Qa}{q_{w,1}(1 + \epsilon)} \quad (86)$$

Of practical engineering interest is the variation of the wall temperatures  $t_{w,1}$  and  $t_{w,2}$  along the length of the duct. This variation is found by evaluating equation (85b) at  $\bar{y} = 1$  and at  $\bar{y} = -1$ , respectively, with the results

$$\frac{t_{w,1} - t_e}{\frac{q_{w,1}(1 + \epsilon)a}{2\kappa}} = (1 + P)\xi + \frac{1}{3} + \sum_{m=1}^{\infty} \bar{C}_{m,e} e^{-\eta_{m,e}^2 \xi} + \frac{1 - \epsilon}{1 + \epsilon} \left( 1 + \sum_{m=0}^{\infty} \bar{C}_{m,o}^* e^{-\eta_{m,o}^2 \xi} \right) \quad (87)$$

$$\frac{t_{w,2} - t_e}{\frac{q_{w,1}(1 + \epsilon)a}{2\kappa}} = (1 + P)\xi + \frac{1}{3} + \sum_{m=1}^{\infty} \bar{C}_{m,e} e^{-\eta_{m,e}^2 \xi} - \frac{1 - \epsilon}{1 + \epsilon} \left( 1 + \sum_{m=0}^{\infty} \bar{C}_{m,o}^* e^{-\eta_{m,o}^2 \xi} \right) \quad (88)$$

Convenient useful alternate forms of equations (87) and (88) are to divide the local wall-to-bulk temperature difference by the fully developed value. The resulting ratio will then approach unity for large distances from the channel entrance. The ratio is formed as follows: The local bulk fluid temperature along the channel is given by the sum



$$t_b(\xi) = t_{Q,b}(\xi) + t_{q,b}(\xi) = t_e + \left[ \frac{(q_{w,1} + q_{w,2})a}{2\kappa} + \frac{Qa^2}{\kappa} \right] \xi \quad (89a)$$

or

$$\frac{t_b(\xi) - t_e}{\frac{q_{w,1}(1 + \epsilon)a}{2\kappa}} = (1 + P)\xi \quad (89b)$$

Then, the difference between the wall and bulk temperatures for each wall is

$$\frac{t_{w,1} - t_b}{\frac{q_{w,1}(1 + \epsilon)a}{2\kappa}} = \frac{1}{3} + \sum_{m=1}^{\infty} \bar{C}_{m,e} e^{-\eta_{m,e}^2 \xi} + \frac{1 - \epsilon}{1 + \epsilon} \left( 1 + \sum_{m=0}^{\infty} \bar{C}_{m,o}^* e^{-\eta_{m,o}^2 \xi} \right) \quad (90)$$

$$\frac{t_{w,2} - t_b}{\frac{q_{w,1}(1 + \epsilon)a}{2\kappa}} = \frac{1}{3} + \sum_{m=1}^{\infty} \bar{C}_{m,e} e^{-\eta_{m,e}^2 \xi} - \frac{1 - \epsilon}{1 + \epsilon} \left( 1 + \sum_{m=0}^{\infty} \bar{C}_{m,o}^* e^{-\eta_{m,o}^2 \xi} \right) \quad (91)$$

It is to be noted that the wall-to-bulk temperature differences are independent of internal heat generation. The fully developed wall-to-bulk temperature differences are obtained as

$$\frac{(t_{w,1} - t_b)_d}{\frac{q_{w,1}(1 + \epsilon)a}{2\kappa}} = \frac{1}{3} + \frac{1 - \epsilon}{1 + \epsilon} \quad (92)$$

and

$$\frac{(t_{w,2} - t_b)_d}{\frac{q_{w,1}(1 + \epsilon)a}{2\kappa}} = \frac{1}{3} - \frac{1 - \epsilon}{1 + \epsilon} \quad (93)$$

The ratios of local to fully developed temperature differences at any location in the channel are found from equations (90) to (93) as

$$\frac{t_{w,1} - t_b}{(t_{w,1} - t_b)_d} = 1 + \frac{\sum_{m=1}^{\infty} \bar{C}_{m,e} e^{-\eta_m^2 e^\xi} + \frac{1-\epsilon}{1+\epsilon} \sum_{m=0}^{\infty} \bar{C}_{m,o}^* e^{-\eta_m^2 o^\xi}}{\frac{1}{3} + \frac{1-\epsilon}{1+\epsilon}} \quad (94)$$

and

$$\frac{t_{w,2} - t_b}{(t_{w,2} - t_b)_d} = 1 + \frac{\sum_{m=1}^{\infty} \bar{C}_{m,e} e^{-\eta_m^2 e^\xi} - \frac{1-\epsilon}{1+\epsilon} \sum_{m=0}^{\infty} \bar{C}_{m,o}^* e^{-\eta_m^2 o^\xi}}{\frac{1}{3} - \frac{1-\epsilon}{1+\epsilon}} \quad (95)$$

To illustrate the results, the bulk temperature developments (eq. (89b)) are plotted in figure 6 for various values of the parameter  $P' = 2Qa/q_{w,1}$  and for several values of the heat flux ratio  $\epsilon = q_{w,2}/q_{w,1}$ . For  $\epsilon = -1$ , the heat addition at the upper wall is equal to the heat extraction at the lower wall. The case for which the lower wall is insulated is represented by  $\epsilon = 0$ , whereas symmetrical heating corresponds to  $\epsilon = 1$ .

As was the case for the tube with internal heat sources and wall heat transfer, a variety of trends are evident in figure 6 that depend on whether the parameter  $P'$  is positive or negative and also on the specific magnitude. These trends occur for reasons similar to those discussed in connection with the round tube; namely, for large values (in magnitude) of  $P'$ , internal heat generation dominates, while for small  $P'$ , wall heat transfer dominates. Further discussion would add little to what has been said in connection with the round tube.

Equations (94) and (95) have been plotted in figure 7 as functions of distance along the channel for parametric values of  $\epsilon$ . A line delineating the condition  $(t_w - t_b)/(t_w - t_b)_d = 0.95$  is shown in the figure to facilitate finding the entrance length. It should be noted that the length required to approach fully developed conditions is least for the duct with symmetrical wall heat flux. The information given on these plots permits rapid evaluation of wall-to-bulk temperature differences at various stations along the duct walls.

The Nusselt number for the upper wall  $Nu_1 \equiv h_1 D_H / \kappa$ , where  $D_H$  is the hydraulic diameter ( $D_H = 4a$  for the parallel-plate channel), may be written as

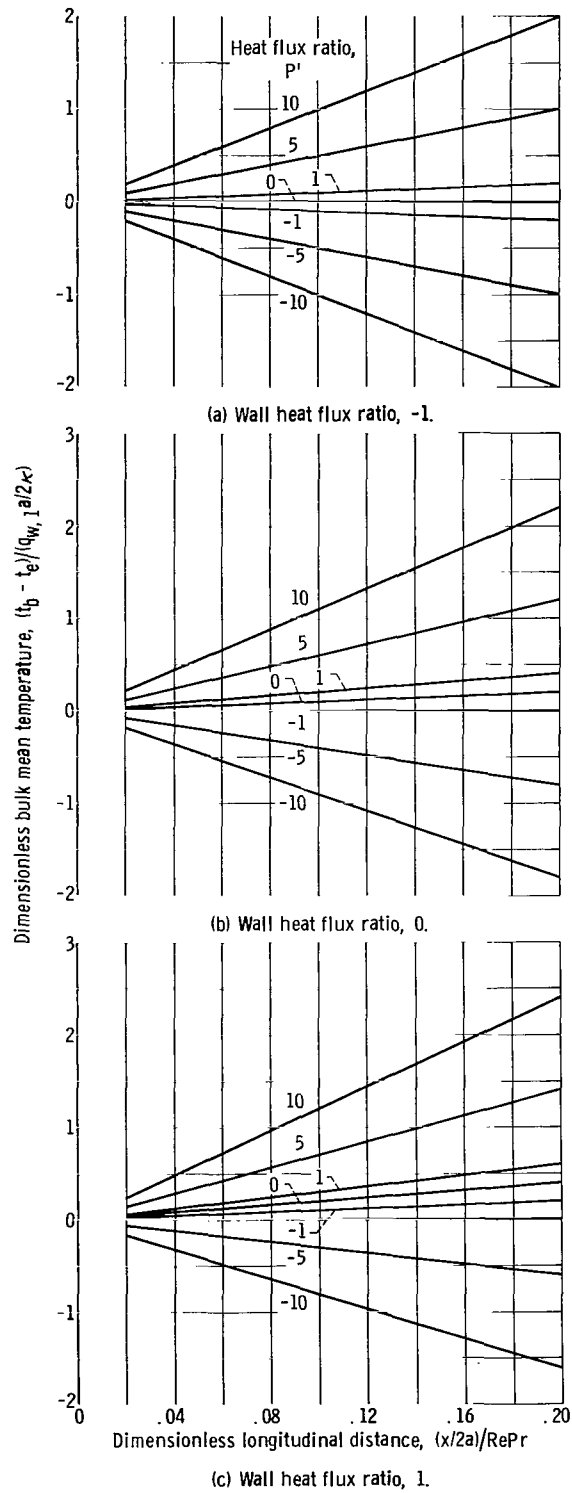


Figure 6. - Bulk mean temperature development for uniform flow in the parallel-plate channel with internal heat generation and wall heat transfer.

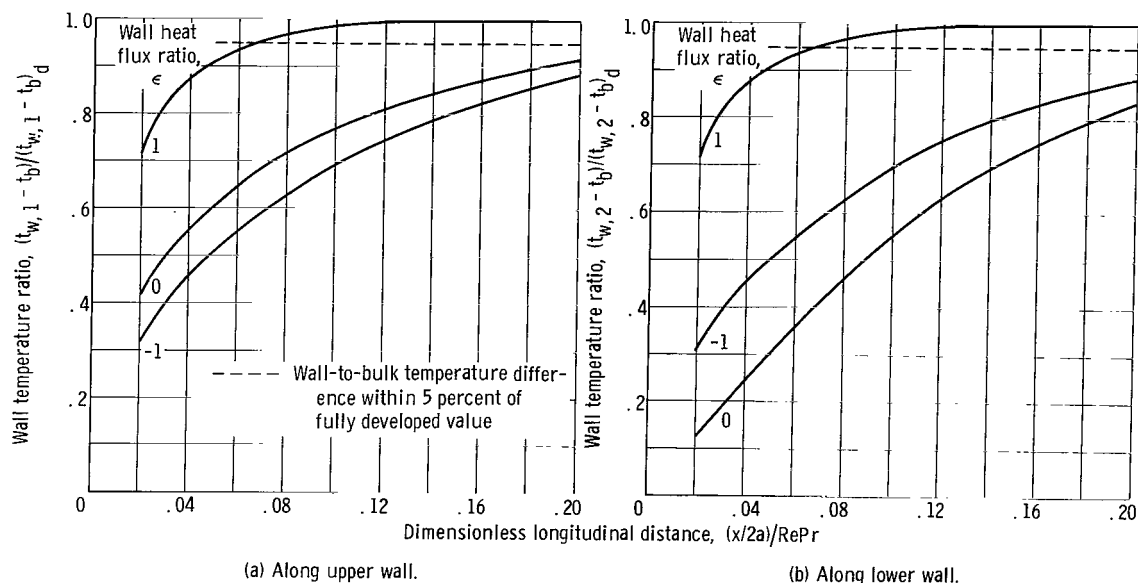


Figure 7. - Wall-to-bulk temperature variation for uniform internal heat generation in a parallel-plate channel with unsymmetrical wall heat transfer.

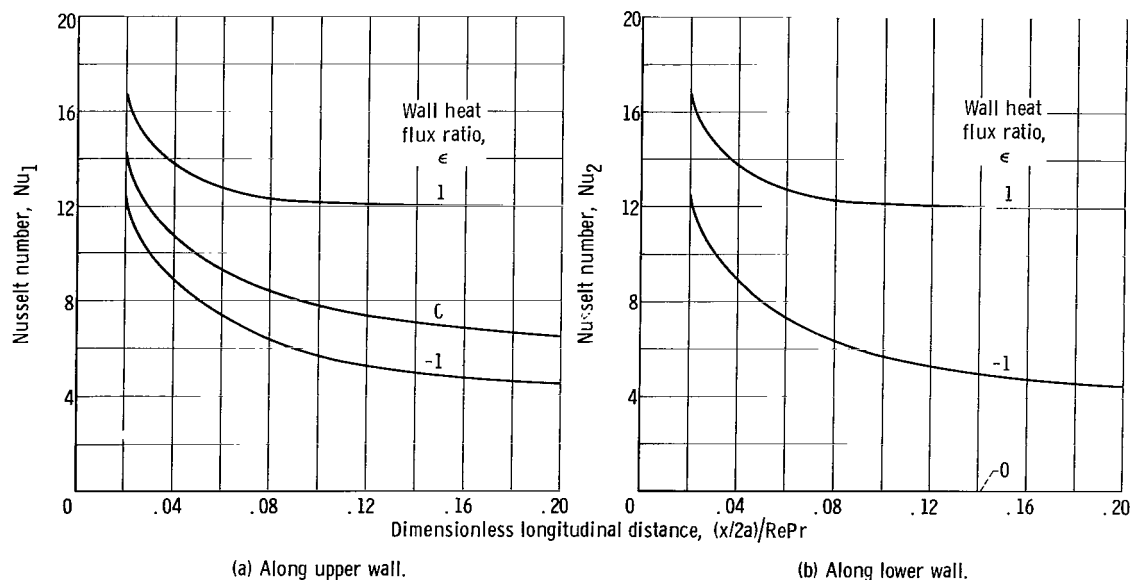


Figure 8. - Nusselt number variation for uniform heat generation in a parallel-plate channel with unsymmetrical wall heat transfer.

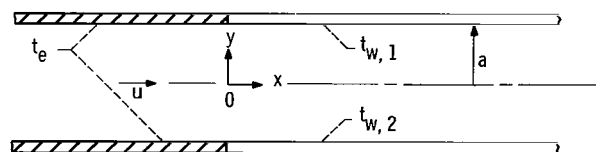


Figure 9. - Physical model and coordinate system for parallel-plate channel with uniform, unequal wall temperatures.

$$\text{Nu}_1 = \frac{q_{w,1}}{t_{w,1} - t_b} \frac{4a}{\kappa} = \frac{8}{1 + \epsilon} \frac{q_{w,1}(1 + \epsilon)a}{2\kappa(t_{w,1} - t_b)} \quad (96)$$

The Nusselt number at the lower wall  $\text{Nu}_2$  may be written in the same manner as

$$\text{Nu}_2 = \frac{q_{w,2}}{t_{w,2} - t_b} \frac{4a}{\kappa} = \frac{8}{1 + \epsilon} \frac{q_{w,2}(1 + \epsilon)a}{2\kappa(t_{w,2} - t_b)} \quad (97)$$

Final expressions for the Nusselt numbers may be written, with the use of equations (90) and (91), as

$$\text{Nu}_1 = \frac{\frac{8}{1 + \epsilon}}{\frac{1}{3} + \sum_{m=1}^{\infty} \bar{C}_{m,e} e^{-\eta_{m,e}^2} + \frac{1 - \epsilon}{1 + \epsilon} \left( 1 + \sum_{m=0}^{\infty} \bar{C}_{m,o}^* e^{-\eta_{m,o}^2} \right)} \quad (98)$$

$$\text{Nu}_2 = \frac{\frac{8\epsilon}{1 + \epsilon}}{\frac{1}{3} + \sum_{m=1}^{\infty} \bar{C}_{m,e} e^{-\eta_{m,e}^2} - \frac{1 - \epsilon}{1 + \epsilon} \left( 1 + \sum_{m=0}^{\infty} \bar{C}_{m,o}^* e^{-\eta_{m,o}^2} \right)} \quad (99)$$

Figure 8 represents Nusselt numbers that have been calculated with equations (98) and (99) for parametric values of  $\epsilon$  of -1, 0, and 1. An inspection of the figure reveals that, in general, the local heat-transfer coefficients are very large near the duct entrance and decrease continuously with increasing downstream distance. For  $\epsilon = 0$ , the Nusselt number for the lower plate is zero, inasmuch as the wall is insulated and therefore adiabatic. The Nusselt number development at either wall takes place most rapidly for symmetrical heating.

## Uniform Heat Generation and Uniform Unequal Wall Temperatures

The problem to be considered here is posed by a system in which a liquid metal flows

isothermally at temperature  $t_e$  up to a point  $x = 0$  (fig. 9). After this point the channel walls are maintained at constant but unequal temperatures, and, in addition, a uniform internal heat generation commences. The special case of equal wall temperatures ( $t_{w,1} = t_{w,2} \equiv t_w$ ) is shown to be a particular solution of the more general problem treated here.

This case uses the dimensionless temperatures  $\tilde{T}_Q(\xi, \bar{y})$  and  $\tilde{T}_T(\xi, \bar{y})$  defined by

$$\frac{t(\xi, \bar{y}) - t_{w,m}}{t_{w,1} - t_{w,m}} = \frac{t_Q}{t_{w,1} - t_{w,m}} + \frac{t_T - t_{w,m}}{t_{w,1} - t_{w,m}} \equiv \tilde{T}_Q + \tilde{T}_T \quad (100)$$

where

$$t_{w,m} = \frac{(t_{w,1} + t_{w,2})}{2} \quad (101)$$

The dimensionless temperature  $\tilde{T}_Q$  and  $\tilde{T}_T$  satisfy the nondimensional differential equations and boundary conditions:

$$\frac{\partial \tilde{T}_Q}{\partial \xi} = \frac{\partial^2 \tilde{T}_Q}{\partial \bar{y}^2} + \frac{Qa^2}{\kappa(t_{w,1} - t_{w,m})} \quad (102a)$$

$$\left. \begin{aligned} \frac{\partial \tilde{T}_Q}{\partial \bar{y}} &= 0 \quad \text{at } \bar{y} = 0 \\ \tilde{T}_Q &= 0 \quad \text{at } \bar{y} = 1 \\ \tilde{T}_Q &= 0 \quad \text{at } \xi = 0 \end{aligned} \right\} \quad (102b)$$

$$\frac{\partial \tilde{T}_T}{\partial \xi} = \frac{\partial^2 \tilde{T}_T}{\partial \bar{y}^2} \quad (103a)$$

$$\left. \begin{aligned} \tilde{T}_T &= -1 \quad \text{at} \quad \bar{y} = -1 \\ \tilde{T}_T &= 1 \quad \text{at} \quad \bar{y} = 1 \\ \tilde{T}_T &= T_{T,e} \quad \text{at} \quad \xi = 0 \end{aligned} \right\} \quad (103b)$$

where

$$T_{T,e} = \frac{t_e - t_{w,m}}{t_{w,1} - t_{w,m}} \quad (104)$$

For each problem the fully developed situation is considered first and then a difference temperature, applicable in the entrance region, is dealt with. These solutions are then superposed. Finally, the result for the general situation is obtained from the superposition of the basic solutions.

Uniform internal heat generation with wall temperatures  $t_{w,1} = t_{w,2} = 0$ . - For large values of  $x$  a fully developed temperature profile  $\tilde{T}_{Q,d}$  exists in the form

$$\tilde{T}_{Q,d} = \frac{Qa^2}{\kappa(t_{w,1} - t_{w,m})} \frac{1}{2} (1 - \bar{y}^2) \quad (105)$$

For the entrance region, use is again made of a difference temperature  $\tilde{T}_Q^*$  defined by

$$\tilde{T}_Q^* = \tilde{T}_Q - \tilde{T}_{Q,d} \quad (106)$$

The solution for  $\tilde{T}_Q^*(\xi, \bar{y})$ , which is applicable in the thermal entrance region, can be shown to have the form

$$\tilde{T}_Q^* = \frac{Qa^2}{\kappa(t_{w,1} - t_{w,m})} \sum_{n=1}^{\infty} d_n Y_n^* e^{-\theta_n^2 \xi} \quad (107)$$

where the  $\theta_n$  and  $Y_n^*(\bar{y}, \theta_n)$  are the eigenvalues and eigenfunctions of the differential equation

$$\frac{d^2 Y_n^*}{d\bar{y}^2} + \theta_n^2 Y_n^* = 0 \quad (108a)$$

with the boundary conditions

$$\frac{dY_n^*}{d\bar{y}} = 0 \quad \text{at} \quad \bar{y} = 0 \quad (108b)$$

$$Y_n^* = 0 \quad \text{at} \quad \bar{y} = 1 \quad (108c)$$

The solution for  $Y_n^*(\bar{y}, \theta_n)$  that satisfies equations (107a) and (108b) is given by

$$Y_n^* = \cos \theta_n \bar{y} \quad (109)$$

where the arbitrary constant that multiplies equation (109) has been set equal to unity. The boundary condition  $Y_n^*(1) = 0$  (eq. (108c)) requires that the eigenvalues  $\theta_n$  be

$$\theta_n = \frac{2n+1}{2} \pi \quad n = 0, 1, 2, \dots \quad (110)$$

The coefficients  $d_n$  are found by applying the boundary condition at  $\xi = 0$ . Then

$$\tilde{T}_Q^*(0, \bar{y}) = \frac{Qa^2}{\kappa(t_w, 1 - t_w, m)} \sum_{n=0}^{\infty} d_n Y_n^* = - \frac{Qa^2}{\kappa(t_w, 1 - t_w, m)} \frac{1}{2} (1 - \bar{y}^2)$$

and from the properties of the Sturm-Liouville system, the coefficients  $d_n$  are given by

$$d_n = - \frac{\frac{1}{2} \int_0^1 (1 - \bar{y}^2) Y_n^* d\bar{y}}{\int_0^1 Y_n^{*2} d\bar{y}} = - \frac{2}{\theta_n^3 \sin \theta_n} = - \frac{2(-1)^n}{\theta_n^3} \quad (111)$$

The complete solution for the temperature  $\tilde{T}_Q(\xi, \bar{y})$  that applies along the entire channel length is obtained by summing equations (105) and (107) with the result



$$\frac{\frac{T_Q(\xi, \bar{y})}{Qa^2}}{\kappa(t_{w,1} - t_{w,m})} = \frac{1}{2} (1 - \bar{y}^2) + \sum_{n=0}^{\infty} d_n Y_n^* e^{-\theta_n^2 \xi} \quad (112)$$

The fluid mixed-mean temperature  $\tilde{T}_{Q,b}$  is obtained from the definition

$$T_{Q,b} = \frac{\int_0^1 U \tilde{T}_Q d\bar{y}}{\int_0^1 U d\bar{y}} = \int_0^1 T_Q d\bar{y} = \frac{Qa^2}{\kappa(t_{w,1} - t_{w,m})} \left( \frac{1}{3} - 2 \sum_{n=1}^{\infty} \frac{e^{-\theta_n^2 \xi}}{\theta_n^4} \right) \quad (113)$$

Unequal wall temperatures without internal heat generation. - The solution for  $\tilde{T}_T$  consists of two parts. The first part is the fully developed temperature profile

$$\tilde{T}_{T,d} \equiv \frac{t_{T,d} - t_{w,m}}{t_{w,1} - t_{w,m}}$$

The second part is

$$\tilde{T}_T \equiv \frac{t_T - t_{T,d}}{t_{w,1} - t_{w,m}}$$

which is an entrance region solution. The temperature  $\tilde{T}_T$  is then given by the sum

$$\tilde{T}_T(\xi, \bar{y}) = \tilde{T}_{T,d}(\bar{y}) + \tilde{T}_T^*(\xi, \bar{y}) \quad (114)$$

For large downstream distances, the fluid temperature is a function of  $\bar{y}$  alone, and equation (103) written for  $\tilde{T}_{T,d}$  reduces therefore to

$$\frac{\partial^2 \tilde{T}_{T,d}}{\partial \bar{y}^2} = 0 \quad (115)$$

The solution to equation (115) satisfying the boundary conditions

$$\tilde{T}_{T,d} = -1 \text{ at } \bar{y} = -1$$

and

$$\tilde{T}_{T,d} = 1 \text{ at } \bar{y} = 1$$

is

$$\tilde{T}_{T,d} = \bar{y} \quad (116)$$

The entrance region temperature  $\tilde{T}_T^*$  must satisfy the governing equation

$$\frac{\partial \tilde{T}_T^*}{\partial \xi} = \frac{\partial^2 \tilde{T}_T^*}{\partial \bar{y}^2} \quad (117a)$$

with boundary conditions

$$\tilde{T}_T^* = 0 \text{ at } \bar{y} = -1 \text{ and } \bar{y} = 1 \quad (117b)$$

$$\tilde{T}_T^* = T_{T,e} - \bar{y} \text{ at } \xi = 0 \quad (117c)$$

By the method of separation of variables, the solution for  $\tilde{T}_T^*$  is obtained as

$$\tilde{T}_T^*(\xi, \bar{y}) = \sum_{n=1}^{\infty} F_n \Gamma_n(\bar{y}, \lambda_n) e^{-\lambda_n^2 \xi} \quad (118)$$

where the  $\lambda_n^2$  and  $\Gamma_n$  are the eigenvalues and eigenfunctions of the Sturm-Liouville system

$$\frac{d^2 \Gamma_n}{d\bar{y}^2} + \lambda_n^2 \Gamma_n = 0 \quad (119a)$$

where

$$\Gamma_n = 0 \text{ at } \bar{y} = -1 \text{ and } \bar{y} = 1 \quad (119b)$$

The coefficients  $F_n$  are determined from the temperature distribution at  $\xi = 0$ ; the  $F_n$  are thus obtained from the result

$$F_n = \frac{T_{T,e} \int_{-1}^1 \Gamma_n d\bar{y}}{\int_{-1}^1 \Gamma_n^2 d\bar{y}} - \frac{\int_{-1}^1 \bar{y} \Gamma_n d\bar{y}}{\int_{-1}^1 \Gamma_n^2 d\bar{y}}$$

If  $\Gamma_n$  is an even function, the second term on the right side vanishes; conversely, if  $\Gamma_n$  is an odd function, the first term disappears. Hence, the constants divide into the two classes given by

$$e_n = \frac{T_{T,e} \int_0^1 Y_n^* d\bar{y}}{\int_0^1 Y_n^{*2} d\bar{y}} \quad (120)$$

$$f_n = - \frac{\int_0^1 \bar{y} Z_n^* d\bar{y}}{\int_0^1 Z_n^{*2} d\bar{y}} \quad (121)$$

where  $Y_n^*(\bar{y}) = Y_n^*(-\bar{y})$  (even function) and  $Z_n^*(\bar{y}) = -Z_n^*(-\bar{y})$  (odd function). The corresponding eigenvalues are denoted by  $\theta_n^2$  and  $\mu_n^2$ . The solution for  $\tilde{T}_T^*$  may then be rewritten as

$$\tilde{T}_T^*(\xi, \bar{y}) = \sum_{n=0}^{\infty} e_n Y_n^* e^{-\theta_n^2 \xi} + \sum_{n=1}^{\infty} f_n Z_n^* e^{-\mu_n^2 \xi} \quad (122)$$

where the eigenvalues  $\theta_n^2$  and  $\mu_n^2$  and corresponding eigenfunctions  $Y_n^*$  and  $Z_n^*$  are solutions to the following differential equations:

$$\frac{d^2 Y_n^*}{d\bar{y}^2} + \theta_n^2 Y_n^* = 0 \quad (123a)$$

$$\frac{dY_n^*}{d\bar{y}} = 0 \quad \text{at } \bar{y} = 0 \quad (123b)$$

$$Y_n^* = 0 \quad \text{at } \bar{y} = 1 \quad (123c)$$

$$\frac{d^2 Z_n^*}{d\bar{y}^2} + \mu_n^2 Z_n^* = 0 \quad (124a)$$

$$Z_n^* = 0 \quad \text{at } \bar{y} = 0 \quad (124b)$$

$$Z_n^* = 0 \quad \text{at } \bar{y} = 1 \quad (124c)$$

The solutions for  $Y_n^*, \theta_n$  are obtained readily from previous results as

$$Y_n^* = \cos \theta_n \bar{y} \quad (125a)$$

$$\theta_n = \frac{2n+1}{2} \pi \quad n = 0, 1, 2, \dots \quad (125b)$$

The solution for  $Z_n^*$  that satisfies equations (124) is given by

$$Z_n^* = \sin \mu_n \bar{y} \quad (126)$$

The boundary condition  $Z_n^*(1) = 0$  requires that the eigenvalues  $\mu_n$  be given by

$$\mu_n = n\pi \quad n = 1, 2, \dots \quad (127)$$

The result for the coefficients  $e_n$ , when divided by the wall-temperature parameter  $T_{T,e}$ , reduces to

$$\frac{e_n}{T_{T,e}} = \frac{\int_0^1 Y_n^* d\bar{y}}{\int_0^1 Y_n^{*2} d\bar{y}} = \frac{2}{\theta_n \sin \theta_n} = \frac{2(-1)^n}{\theta_n} \quad (128)$$

The coefficients  $e_n$  may therefore be evaluated for a given value of  $T_{T,e}$  - for example,  $T_{T,e} = 1$ . For other values of  $T_{T,e}$ , it is only necessary to multiply the  $e_n$  for  $T_{T,e} = 1$  by the new value of  $T_{T,e}$  to obtain the new coefficients.

When equations (121) and (126) are used, the coefficients  $f_n$  are obtained readily as

$$f_n = \frac{2}{\mu_n \cos \mu_n} = \frac{2(-1)^n}{\mu_n} \quad (129)$$

The complete solution for the temperature  $\tilde{T}_T(\xi, \bar{y})$  that applies along the entire channel length is obtained by summing equations (116) and (122) with the result

$$\tilde{T}_T(\xi, \bar{y}) = \bar{y} + \sum_{n=0}^{\infty} e_n Y_n^* e^{-\theta_n^2 \xi} + \sum_{n=1}^{\infty} f_n Z_n^* e^{-\mu_n^2 \xi} \quad (130)$$

The fluid mixed-mean temperature  $t_{T,b}$  is obtained from the definition

$$t_{T,b} = \frac{1}{2} \int_{-1}^1 t_T d\bar{y} \quad (131a)$$

Equation (131a) may be written in dimensionless form as

$$\frac{t_{T,b} - t_{w,m}}{t_{w,1} - t_{w,m}} \equiv \tilde{T}_{T,b} = \frac{1}{2} \int_{-1}^1 \tilde{T}_T d\bar{y} \quad (131b)$$

When equation (130) is substituted for  $\tilde{T}_T$  and the integration is carried out, the bulk temperature  $\tilde{T}_{T,b}$  is determined as

$$\tilde{T}_{T,b} = \sum_{n=1}^{\infty} \frac{2T_{T,e}}{\theta_n^2} e^{-\theta_n^2 \xi} \quad (132)$$

Internal heat generation with unequal wall temperatures. - The separate results presented previously may be combined to provide results for the situation where internal heat generation  $Q$  and temperature driving force  $t_{w,1} - t_{w,m} = t_{w,1} - t_{w,2}$  act simultaneously to give

$$\begin{aligned} \tilde{T} = \frac{t - t_{w,m}}{t_{w,1} - t_{w,m}} = \frac{Qa^2}{\kappa(t_{w,1} - t_{w,m})} & \left[ \frac{1}{2} (1 - \bar{y}^2) + \sum_{n=0}^{\infty} d_n Y_n^* e^{-\theta_n^2 \xi} \right] \\ & + \bar{y} + \sum_{n=0}^{\infty} e_n Y_n^* e^{-\theta_n^2 \xi} + \sum_{n=1}^{\infty} f_n Z_n^* e^{-\theta_n^2 \xi} \end{aligned} \quad (133)$$

It is illuminating to recast this equation into an equivalent form:

$$\begin{aligned} \frac{t - t_{w,m}}{t_e - t_{w,m}} = \frac{Qa^2}{\kappa(t_e - t_{w,m})} & \left[ \frac{1}{2} (1 - \bar{y}^2) - \sum_{n=0}^{\infty} \frac{2(-1)^n}{\theta_n^3} \cos(\theta_n \bar{y}) e^{-\theta_n^2 \xi} \right] \\ & + \sum_{n=0}^{\infty} \frac{2(-1)^n}{\theta_n} \cos(\theta_n \bar{y}) e^{-\theta_n^2 \xi} \\ & + \frac{t_{w,1} - t_{w,2}}{t_e - t_{w,m}} \left[ \bar{y} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{\mu_n} \sin(\mu_n \bar{y}) e^{-\mu_n^2 \xi} \right] \end{aligned} \quad (134)$$

The solution for the special case of equal wall temperatures ( $t_{w,1} = t_{w,2} \equiv t_w$ ) is therefore obtained from equation (134) as

$$\begin{aligned} \frac{t - t_w}{t_e - t_w} = \frac{Qa^2}{\kappa(t_e - t_w)} & \left[ \frac{1}{2} (1 - \bar{y}^2) - \sum_{n=0}^{\infty} \frac{2(-1)^n}{\theta_n^3} \cos(\theta_n \bar{y}) e^{-\theta_n^2 \xi} \right] \\ & + \sum_{n=0}^{\infty} \frac{2(-1)^n}{\theta_n} \cos(\theta_n \bar{y}) e^{-\theta_n^2 \xi} \end{aligned} \quad (135)$$

With the duct wall temperatures specified and the temperature distribution determined, the wall heat flux variations required to maintain the wall temperatures constant are obtained in dimensionless forms as

$$\frac{q_1 a}{\kappa(t_{w,1} - t_{w,m})} = 3N_2 \left( 1 - 2 \sum_{n=0}^{\infty} \frac{e^{-\theta_n^2 \xi}}{\theta_n^2} \right) - 1 + 2T_{T,e} \sum_{n=0}^{\infty} e^{-\theta_n^2 \xi} - 2 \sum_{n=1}^{\infty} e^{-\mu_n^2 \xi} \quad (136)$$

$$\frac{q_2 a}{\kappa(t_{w,1} - t_{w,m})} = 3N_2 \left( -1 + 2 \sum_{n=0}^{\infty} \frac{e^{-\theta_n^2 \xi}}{\theta_n^2} \right) - 1 - 2T_{T,e} \sum_{n=0}^{\infty} e^{-\theta_n^2 \xi} - 2 \sum_{n=1}^{\infty} e^{-\mu_n^2 \xi} \quad (137)$$

where  $N_2 \equiv Qa^2/3\kappa(t_{w,1} - t_{w,m})$  and the results

$$\left( \frac{dY_n^*}{d\bar{y}} \right)_{\bar{y}=1} = - \left( \frac{dY_n^*}{d\bar{y}} \right)_{\bar{y}=-1}$$

and

$$\left( \frac{dZ_n^*}{d\bar{y}} \right)_{\bar{y}=1} = \left( \frac{dZ_n^*}{d\bar{y}} \right)_{\bar{y}=-1}$$

have been used.

For the special case of equal wall temperatures ( $t_{w,1} = t_{w,2} = t_w$ ), the heat-transfer rate  $q$  at the duct wall is given by

$$\frac{qa}{\kappa(t_e - t_w)} = \frac{Qa^2}{\kappa(t_e - t_w)} \left( 1 - 2 \sum_{n=0}^{\infty} \frac{e^{-\theta_n^2 \xi}}{\theta_n^2} \right) + 2 \sum_{n=0}^{\infty} e^{-\theta_n^2 \xi} = 3N_3 \left( 1 - 2 \sum_{n=0}^{\infty} \frac{e^{-\theta_n^2 \xi}}{\theta_n^2} \right) + 2 \sum_{n=0}^{\infty} e^{-\theta_n^2 \xi} \quad (138)$$

where  $N_3 = Qa^2/3\kappa(t_e - t_w)$

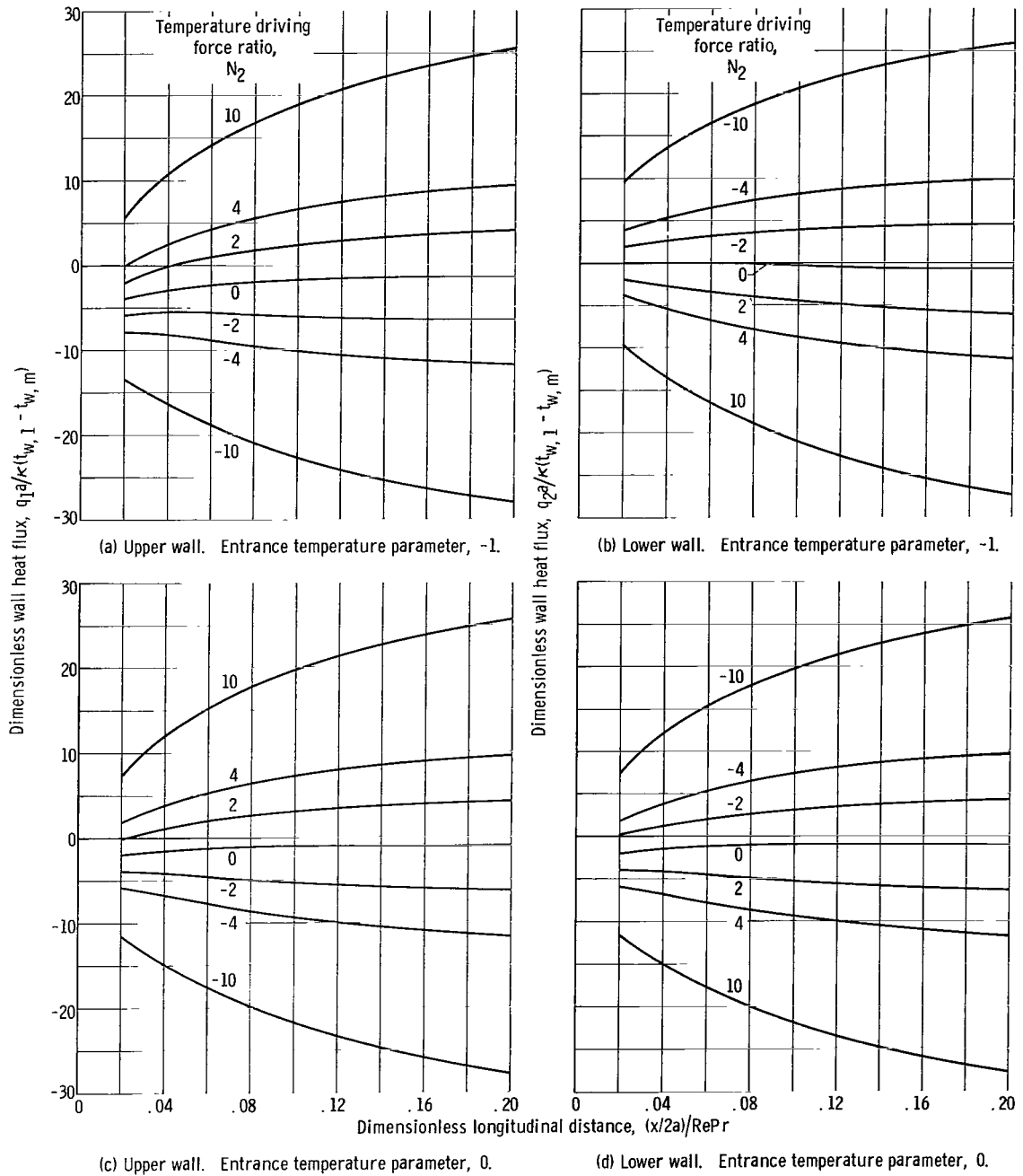


Figure 10. - Heat flux variations for uniform flow in a parallel-plate channel with internal heat generation and unequal wall temperatures.



Illustrative heat-transfer results have been evaluated from equations (136) and (137) for values of  $N_2 = Qa^2/3\kappa(t_{w,1} - t_{w,m})$  ranging from -10 to 10. These quantities also depend on the magnitude of the entrance temperature in relation to the values of the wall temperatures, or on the wall-temperature parameter  $T_{T,e}$ .

The particular values of the wall-temperature parameter  $T_{T,e}$  chosen for the computations correspond to the following cases: (1)  $T_{T,e} = \mp 1$ , in which the lower or upper wall, respectively, is at the temperature of the fluid entering the channel, and (2)  $T_{T,e} = 0$ , in which the arithmetic average of the wall temperatures is equal to the temperature of the fluid entering the channel (i. e.,  $(t_{w,1} + t_{w,2})/2 = t_e$ ).

The foregoing information is given in figure 10 as a function of the longitudinal position coordinate and for the values of the wall-temperature parameter  $T_{T,e}$  of -1 and 0. The results for  $T_{T,e} = 1$  are obtained from the figures for  $T_{T,e} = -1$  by (1) interchanging the subscripts 1 and 2 on the dimensionless wall heat fluxes and (2) interchanging the sign on the parameter  $N_2$  (e. g.,  $q_2a/\kappa(t_{w,1} - t_{w,m})$  for  $T_{T,e} = 1$  and  $N_2 = -10$  is equal to  $q_1a/\kappa(t_{w,1} - t_{w,m})$  for  $T_{T,e} = -1$  and  $N_2 = 10$ ). The variation in wall heat flux required to maintain the channel wall temperatures constant at  $t_w$  is shown in figure 11 for parametric values of the parameter

$N_3$  ranging from -10 to 10. The results closely parallel those that have already been described for the case of liquid-metal flow in a round tube with prescribed surface temperature, so there is no apparent need for repetitive discussion here.

The longitudinal variation of the dimensionless wall-to-bulk temperature difference for each wall is given by

$$\frac{t_{w,1} - t_b}{(t_{w,1} - t_b)_d} = 1 - \frac{6N_2}{N_2 - 1} \sum_{n=0}^{\infty} \frac{e^{-\theta_n^2 \xi}}{\theta_n^4} + \frac{2T_{T,e}}{N_2 - 1} \sum_{n=0}^{\infty} \frac{e^{-\theta_n^2 \xi}}{\theta_n^2} \quad (139)$$

$$\frac{t_{w,2} - t_b}{(t_{w,2} - t_b)_d} = 1 - \frac{6N_2}{N_2 + 1} \sum_{n=0}^{\infty} \frac{e^{-\theta_n^2 \xi}}{\theta_n^4} + \frac{2T_{T,e}}{N_2 + 1} \sum_{n=0}^{\infty} \frac{e^{-\theta_n^2 \xi}}{\theta_n^2} \quad (140)$$

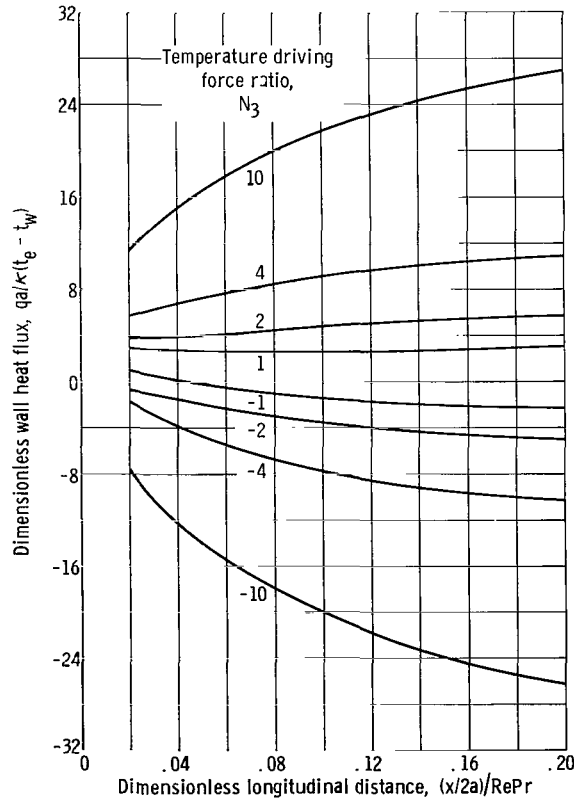


Figure 11. - Heat flux variation for uniform flow in a parallel-plate channel with internal heat generation and equal wall temperatures.

where

$$\frac{(t_b - t_{w,1})_d}{t_{w,1} - t_{w,m}} = N_2 - 1 \quad (141a)$$

$$\frac{(t_b - t_{w,2})_d}{t_{w,1} - t_{w,m}} = N_2 + 1 \quad (141b)$$

$$\frac{t_b - t_{w,1}}{t_{w,1} - t_{w,m}} = \tilde{T}_{Q,b} + \tilde{T}_{T,b} - 1 = \tilde{T}_b - 1 \quad (141c)$$

and

$$\frac{t_b - t_{w,2}}{t_{w,1} - t_{w,m}} = \tilde{T}_b + 1 \quad (141d)$$

The variation of the dimensionless wall-to-bulk temperature difference along the channel for the special case  $t_{w,1} = t_{w,2} = t_w$  is obtained as

$$\frac{t_w - t_b}{(t_w - t_b)_d} = 1 - 6 \sum_{n=0}^{\infty} \frac{e^{-\theta_n^2 \xi}}{\theta_n^4} + N_3 \sum_{n=0}^{\infty} \frac{e^{-\theta_n^2 \xi}}{\theta_n^2} \quad (142)$$

$$\frac{(t_b - t_w)_d}{t_e - t_w} = \frac{Qa^2}{3\kappa(t_e - t_w)} = N_3 \quad (143)$$

and

$$\frac{t_b - t_w}{t_e - t_w} = T_{Q,b} + T_{T,b} \quad (144)$$

Results for the wall-to-bulk temperature differences that apply along the length of the channel are plotted on figure 12 as a function of the longitudinal position coordinate for

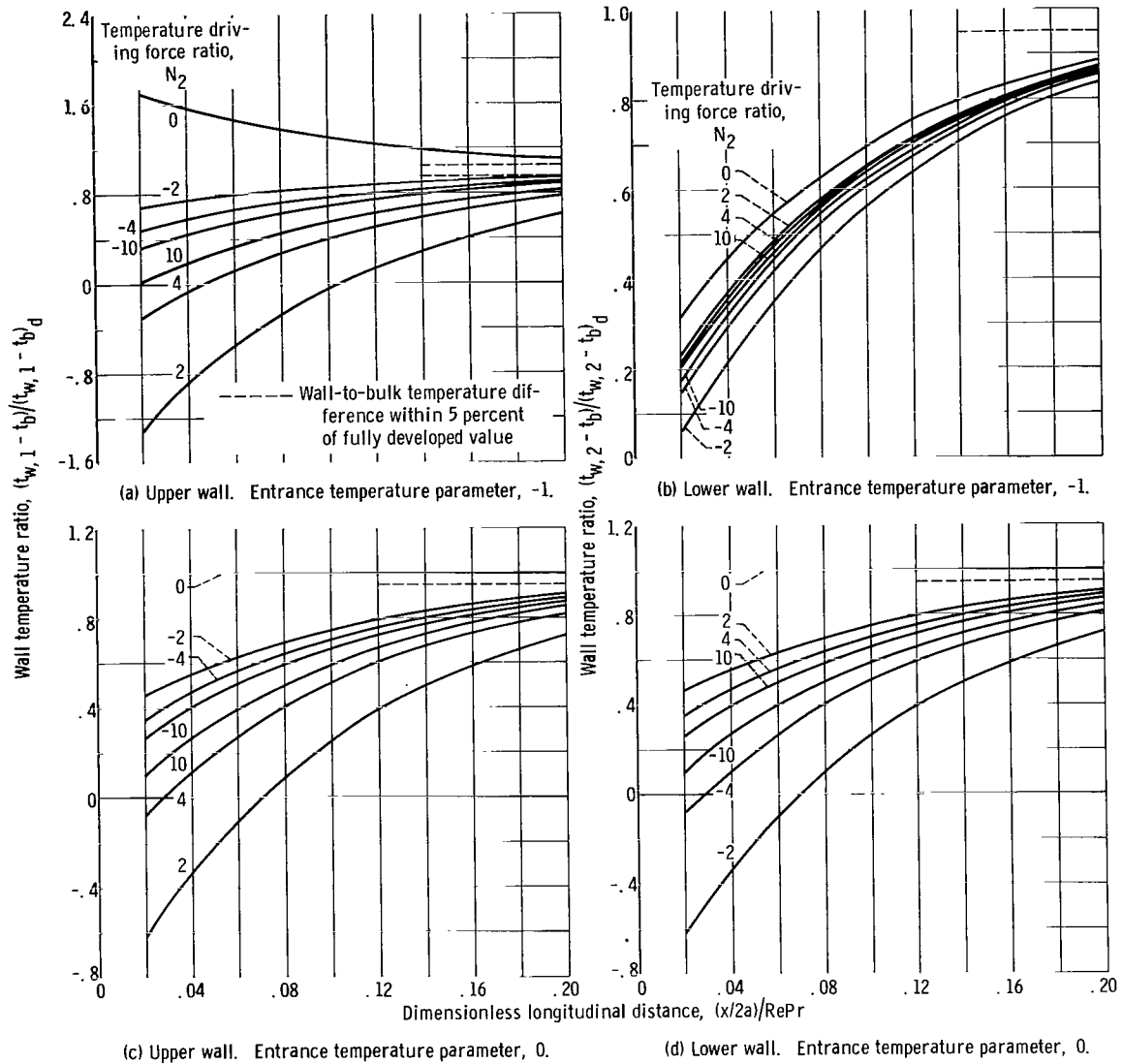


Figure 12. - Wall temperature ratio for uniform flow in a parallel-plate channel with internal heat generation and unequal wall temperatures.

parametric values of  $N_2$  and for  $T_{T,e}$  values of -1 and 0. The results for  $T_{T,e} = 1$  are obtained from the figures for  $T_{T,e} = -1$  by (1) interchanging the subscripts 1 and 2 on the wall temperature  $t_w$  and (2) interchanging the sign on the parameter  $N_2$ . A dashed horizontal line has been drawn in the figure to aid in determining the effect of the parameter  $N_2$  on the thermal entrance length for each wall that is defined, as before, as the length required for  $t_w - t_b$  to approach within 5 percent of its fully developed value. Figure 12 shows that the thermal entrance length for either wall may be appreciably affected by the unsymmetrical boundary condition and the relative magnitude of internal heat generation to that of wall-temperature driving force. For  $T_{T,e} = 0$  and  $N_2 = 0$ , the

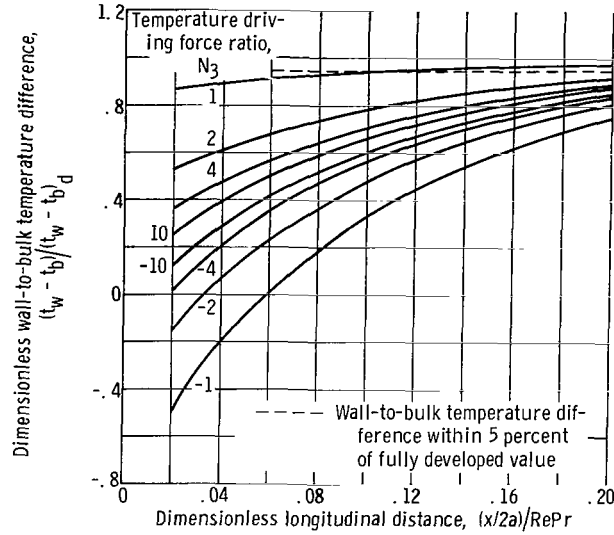


Figure 13. - Wall temperature ratio for uniform flow in a parallel-plate channel with internal heat generation and equal wall temperatures.

wall temperature ratio at either wall assumes the value of unity and remains constant for  $x \geq 0$ .

Equation (142) has been plotted in figure 13 for several values of the parameter  $N_3$ . The trends are similar to those observed in the circular tube problem.

The Nusselt number  $Nu_1$  for the upper wall may be written

$$Nu_1 = \frac{q_1}{t_b - t_{w,1}} \frac{4a}{\kappa} = 4 \frac{q_1 a}{\kappa(t_{w,1} - t_{w,m})} \frac{t_{w,1} - t_{w,m}}{t_b - t_{w,1}} \quad (145)$$

and hence

$$Nu_1 = 4 \frac{q_1 a}{\kappa(t_{w,1} - t_{w,m})} \frac{1}{\tilde{T}_b - 1} \quad (146)$$

The Nusselt number  $Nu_2$  at the lower wall may be written in the same manner as

$$Nu_2 = \frac{q_2}{t_{w,2} - t_b} \frac{4a}{\kappa} = -4 \frac{q_2 a}{\kappa(t_{w,1} - t_{w,m})} \frac{1}{\tilde{T}_b + 1} \quad (147)$$

Final expressions for the Nusselt numbers may be written, with the use of equations (113), (132), (136), (137), (146), and (147), as

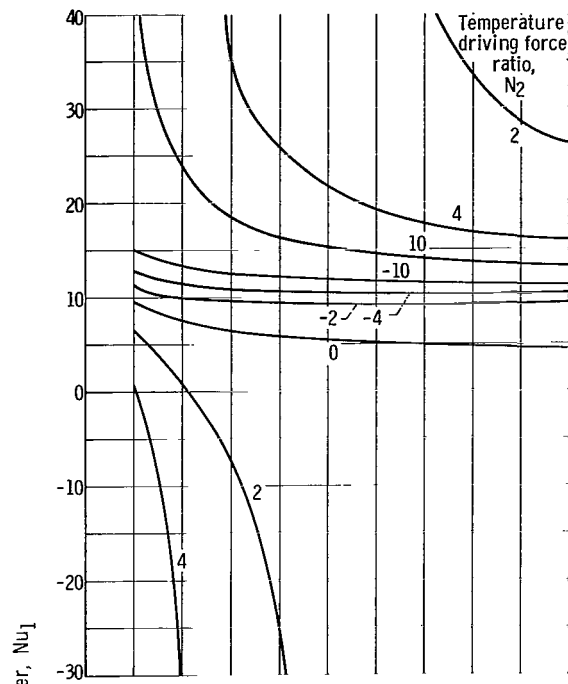
$$\text{Nu}_1 = \frac{12N_2 \left( 1 - 2 \sum_{n=0}^{\infty} \frac{e^{-\theta_n^2 \xi}}{\theta_n^2} \right) + 8T_{T,e} \sum_{n=0}^{\infty} e^{-\theta_n^2 \xi} - 4 \left( 1 + 2 \sum_{n=1}^{\infty} e^{-\mu_n^2 \xi} \right)}{N_2 \left( 1 - 2 \sum_{n=0}^{\infty} \frac{e^{-\theta_n^2 \xi}}{\theta_n^4} \right) + 2T_{T,e} \sum_{n=0}^{\infty} \frac{e^{-\theta_n^2 \xi}}{\theta_n^2} - 1} \quad (148)$$

$$\text{Nu}_2 = \frac{12N_2 \left( 1 - 2 \sum_{n=0}^{\infty} \frac{e^{-\theta_n^2 \xi}}{\theta_n^2} \right) + 8T_{T,e} \sum_{n=0}^{\infty} e^{-\theta_n^2 \xi} + 4 \left( 1 + 2 \sum_{n=1}^{\infty} e^{-\mu_n^2 \xi} \right)}{N_2 \left( 1 - 2 \sum_{n=0}^{\infty} \frac{e^{-\theta_n^2 \xi}}{\theta_n^4} \right) + 2T_{T,e} \sum_{n=0}^{\infty} \frac{e^{-\theta_n^2 \xi}}{\theta_n^2} + 1} \quad (149)$$

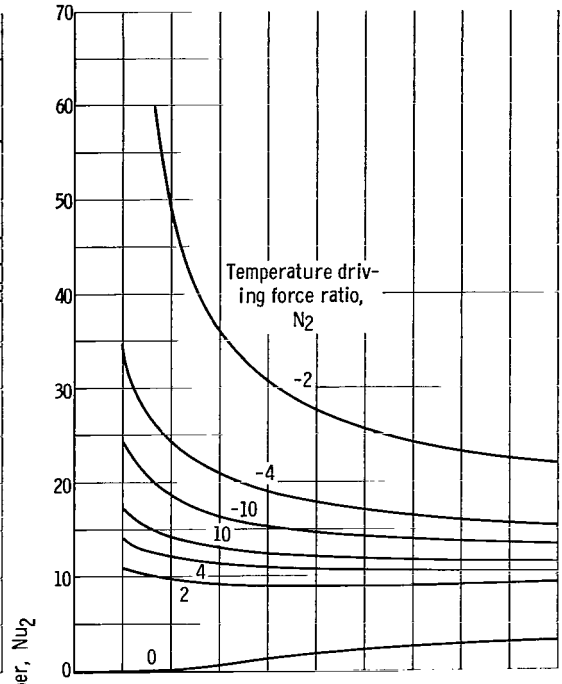
The Nusselt number for the special case  $t_{w,1} = t_{w,2} = t_w$  may be determined from the definition

$$\text{Nu} \equiv \frac{q}{t_b - t_w} \frac{4a}{\kappa} = 4 \frac{qa}{\kappa(t_e - t_w)} \frac{t_e - t_w}{t_b - t_w}$$

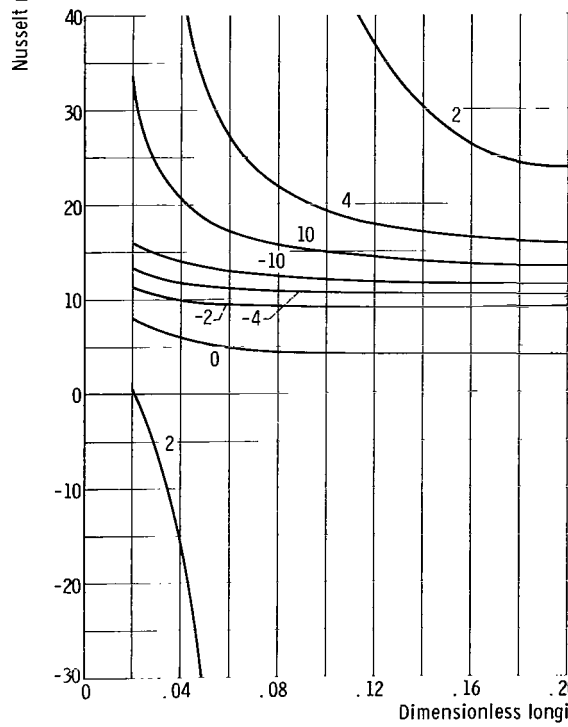
Using equations (138), (142), and (143) yields



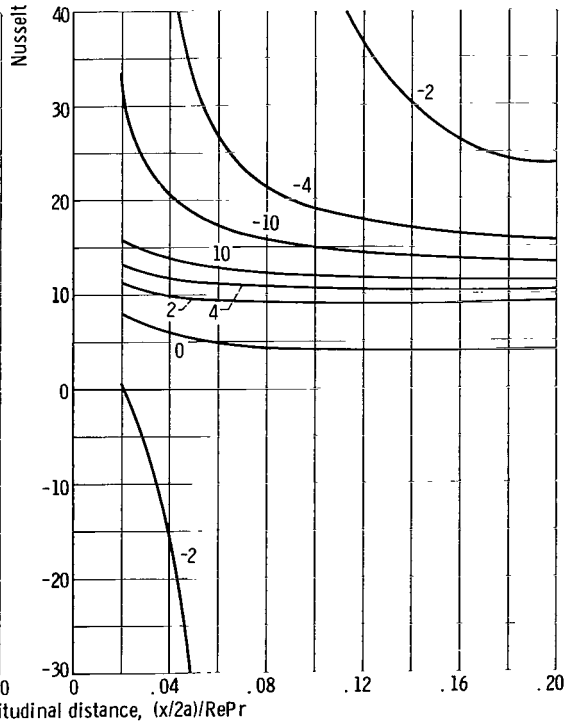
(a) Upper wall. Entrance temperature parameter, -1.



(b) Lower wall. Entrance temperature parameter, -1.



(c) Upper wall. Entrance temperature parameter, 0.



(d) Lower wall. Entrance temperature parameter, 0.

Figure 14. - Nusselt number variation for uniform flow in a parallel-plate channel with internal heat generation and unequal wall temperatures.

$$Nu = \frac{12N_3 \left( 1 - 2 \sum_{n=0}^{\infty} \frac{e^{-\theta_n^2 \xi}}{\theta_n^2} \right) + 8 \sum_{n=0}^{\infty} e^{-\theta_n^2 \xi}}{N_3 \left( 1 - 6 \sum_{n=0}^{\infty} \frac{e^{-\theta_n^2 \xi}}{\theta_n^4} \right) + 2 \sum_{n=0}^{\infty} \frac{e^{-\theta_n^2 \xi}}{\theta_n^2}} \quad (150)$$

Variations of the local Nusselt numbers as functions of the longitudinal position along the length of the channel for several parametric values of  $N_2$  are shown on figure 14 for  $T_{T,e}$  values of -1 and 0. The results for  $T_{T,e} = 1$  are obtained from the figures for  $T_{T,e} = -1$  by interchanging the subscripts 1 and 2 on the Nusselt number and reversing the sign on the parameter  $N_2$ . The results show that the unsymmetrical boundary condition and the magnitude of the parameter  $N_2$  have an appreciable effect on the Nusselt numbers.

The local Nusselt number for the special case  $t_{w,1} = t_{w,2} = t_w$  has been evaluated as a function of the longitudinal position coordinate, and the results are plotted in figure 15 with  $N_3$  appearing as a parameter.

As was the case for flow in a round tube with prescribed wall temperature, infinities and zeroes occur in the Nusselt numbers; this situation corresponds to the point at which the bulk temperature passes the particular wall temperature, and hence, the temperature difference  $t_w - t_b$  changes sign. The heat-transfer coefficient has little meaning for this situation.

For nonzero values of  $N_2$  the following is obtained for the fully developed ( $\xi \rightarrow \infty$ ) Nusselt numbers  $Nu_{1,d}$  and  $Nu_{2,d}$ :

$$Nu_{1,d} = \frac{12N_2 - 4}{N_2 - 1}$$

$$Nu_{2,d} = \frac{12N_2 + 4}{N_2 + 1}$$

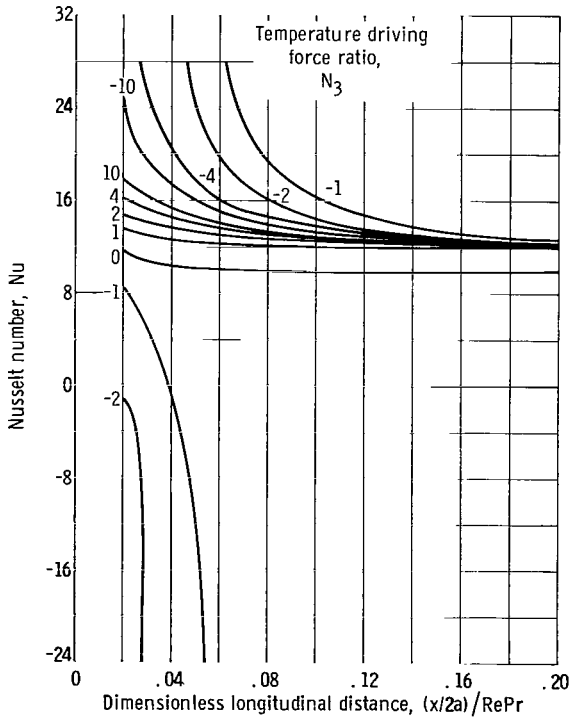


Figure 15. - Nusselt number variation for uniform flow in a parallel-plate channel with internal heat generation and equal wall temperatures.

These asymptotic values depend on the magnitude of the parameter  $N_2$ .

For nonzero values of  $N_3$ , the fully developed Nusselt number  $Nu_d$  for a channel with equal wall temperatures is given by

$$Nu_d = 12 \quad \xi \rightarrow \infty$$

The asymptotic value therefore is independent of the parameter  $N_3$ . When  $N_3 = 0$ , the Nusselt number variation is obtained from equation (150) as

$$Nu = 4 \frac{\sum_{n=0}^{\infty} e^{-\theta_n^2 \xi}}{\sum_{n=0}^{\infty} \frac{1}{\theta_n^2} e^{-\theta_n^2 \xi}}$$

The fully developed heat-transfer condition occurs for  $\xi$  sufficiently large so that only the first term of the series need be considered. The fully developed Nusselt number for this situation then follows as

$$Nu_d = 4\theta_0^2 = 9.87$$

## CONCLUDING REMARKS

Heat transfer to a liquid metal flowing in a round tube or a flat duct with heat sources distributed uniformly in the fluid has been studied analytically. The various effects considered are believed to have increased the knowledge of the forced-convection heat-transfer characteristics for turbulent liquid-metal flow in circular tubes and flat ducts with internal heat generation.

The simplification provided by the slug-flow and negligible thermal-eddy-diffusivity assumptions has made it possible to obtain exact mathematical solutions to the governing energy equation. From these it is possible to gain a physical interpretation of the heat-transfer characteristics for different thermal boundary conditions and internal heat generation. The conclusions that may be drawn from the present study are as follows:



1. The influence of an internal heat source on the thermal entrance length and on the local heat-transfer coefficient (or Nusselt number) is expected to be slight for liquid-metal flow in ducts with prescribed wall heat flux.

2. The thermal entrance length and heat-transfer coefficient for liquid-metal flow in a duct with prescribed wall temperatures may be appreciably effected by the presence of an internal heat source in the fluid.

3. Limiting solutions (e. g. , asymptotic Nusselt numbers for heat transfer to slug flows in pipes or between parallel plates with uniform heat flux and no internal heat generation) are in agreement with published slug-flow solutions (ref. 28) and in qualitative agreement with recommended equations (e. g. , ref. 29).

4. The postulate that the eddy diffusivity of heat is small compared with molecular diffusivity would seem to apply well for liquid metals flowing at moderate velocities; at high velocities, however, the fact that eddy conduction is neglected will probably cause inaccuracies in the numerical values.

The model considered here cannot be expected to yield quantitative results that will prove adequate for all circumstances since the model represents a first approximation to the problem of heat transfer to turbulent liquid-metal flow in conduits with heat generation in the fluid.

For improved heat-transfer calculations, velocity profile and eddy diffusivity variations would have to be taken into account to provide a somewhat more realistic description. Turbulent velocity profiles for pipe and parallel-plate duct systems can be satisfactorily represented by a seventh power law. Eddy diffusivity variations have been given by Poppendiek (ref. 30) and have been used to predict heat transfer in turbulent flow in flat ducts.

The range of validity of the present solutions must be established primarily by comparison with experimental data. Within the knowledge of the author, there are no published experimental results for entrance-region heat transfer to liquid metals flowing turbulently in conduits with wall heat transfer or constant wall temperature and internal heat generation in the fluid with which to make such a comparison.

It is expected that the graphs and tables presented in this report will be helpful to the engineer in selecting a preliminary reference design or in estimating heat-transfer performance in an existing design for a liquid-metal system.

Lewis Research Center,  
National Aeronautics and Space Administration,  
Cleveland, Ohio, March 30, 1966.

## APPENDIX - SYMBOLS

$A_j$	coefficient in series for temperature distribution in circular tube with wall heat transfer and $Q = 0$
$\bar{A}_j$	coefficient defined by eq. (16)
$a$	half distance between parallel plates
$B_k$	coefficient in series for temperature distribution in circular tube with uniform wall temperature and $Q = 0$
$\bar{B}_k$	coefficient defined by eq. (40)
$b_k$	coefficient in series for temperature distribution in circular tube with internal heat generation and $t_w = 0$
$\bar{b}_k$	coefficient defined by eq. (33)
$C_m$	coefficient in series for temperature distribution in parallel-plate channel with wall heat transfer and $Q = 0$
$C_{m,e}$	even coefficient in series for temperature distribution in parallel-plate channel with wall heat transfer and $Q = 0$
$\bar{C}_{m,e}$	coefficient defined by eq. (80)
$C_{m,o}$	odd coefficient in series for temperature distribution in parallel-plate channel with wall heat transfer and $Q = 0$
$C_{m,o}^*$	odd coefficient divided by $(1 - \epsilon)/(1 + \epsilon)$ (i. e., $C_{m,o}/[(1 - \epsilon)/(1 + \epsilon)]$ )
$\bar{C}_{m,o}^*$	coefficient defined by eq. (82)
$c_p$	specific heat of fluid at constant pressure
$D$	diameter of tube, $2r_0$
$D_H$	hydraulic diameter of duct, $4(\text{cross-sectional area of duct})/(\text{wetted perimeter})$
$d_n$	coefficient in series for temperature distribution in parallel-plate channel with internal heat generation and $t_{w,1} = t_{w,2} = 0$
$e_n$	even coefficient in series for temperature distribution in parallel-plate channel with uniform unequal wall temperatures and $Q = 0$
$F_n$	coefficient in series for temperature distribution in parallel-plate channel with uniform unequal wall temperatures and $Q = 0$
$f_n$	odd coefficient in series for temperature distribution in parallel-plate channel with uniform unequal wall temperatures and $Q = 0$

$h$	heat-transfer coefficient, $q/(t_w - t_b)$
$J_0$	Bessel function of order zero of first kind
$J_1$	Bessel function of order one of first kind
$M$	heat flux ratio, $Qr_0/2q_w$
$Nu$	Nusselt number, $hD_H/\kappa$ or $hD/\kappa$
$N_1$	temperature driving force ratio, $Qr_0^2/8\kappa(t_e - t_w)$
$N_2$	temperature driving force ratio, $Qa^2/3\kappa(t_{w,1} - t_{w,m})$
$N_3$	temperature driving force ratio, $Qa^2/3\kappa(t_e - t_w)$
$P$	heat flux ratio, $2Qa/q_{w,1}(1 + \epsilon)$
$P'$	heat flux ratio, $2Qa/q_{w,1}$
$Pe$	Péclet number
$Pr$	Prandtl number, $\mu c_p/\kappa$
$Q$	rate of internal heat generation per unit volume
$q$	rate of heat transfer per unit area from liquid to wall for tube or channel with uniform wall temperature
$q_w$	rate of heat transfer per unit area from wall to fluid for tube or channel with uniform wall heat transfer
$q_{w,1}$	rate of heat transfer per unit area from upper wall to fluid for channel with uniform unequal wall heat fluxes
$q_{w,2}$	rate of heat transfer per unit area from lower wall to fluid for channel with uniform unequal wall heat fluxes
$q_1$	rate of heat transfer per unit area from fluid to upper wall for channel with uniform unequal wall temperatures
$q_2$	rate of heat transfer per unit area from lower wall to liquid for channel with uniform unequal wall temperatures
$R_j(\bar{r}, \beta_j)$	eigenfunctions for tube flow with wall heat transfer and $Q = 0$
$\tilde{R}_k(\bar{r}, \gamma_k)$	eigenfunctions for tube flow with uniform wall temperature or uniform inter- nal heat generation
$Re$	Reynolds number, $2\rho Ua/\mu$ or $\rho UD/\mu$
$r$	radial coordinate
$\bar{r}$	dimensionless radial coordinate, $\bar{r}/r_0$

$r_0$	tube radius
$T$	dimensionless temperature, $[t(\xi, \bar{r}) - t_w]/(t_e - t_w)$
$T_Q$	dimensionless temperature, $t_Q(\xi, \bar{r})/(t_e - t_w)$
$T_T$	dimensionless temperature, $[t_T(\xi, \bar{r}) - t_w]/(t_e - t_w)$
$\tilde{T}$	dimensionless temperature, $(t - t_{w, m})/(t_{w, 1} - t_{w, m})$
$\tilde{T}_Q$	dimensionless temperature, $t_Q(\xi, \bar{y})/(t_{w, 1} - t_{w, m})$
$\tilde{T}_Q^*$	dimensionless entrance region temperature, $t_Q^*/(t_{w, 1} - t_{w, m})$
$\tilde{T}_T$	dimensionless temperature, $[t_T(\xi, \bar{y}) - t_{w, m}]/(t_{w, 1} - t_{w, m})$
$\tilde{T}_T^*$	dimensionless entrance region temperature, $(t_T^* - t_{w, m})/(t_{w, 1} - t_{w, m})$
$t$	fluid temperature
$t_Q$	fluid temperature in presence of internal heat generation
$t_Q^*$	entrance region temperature in the presence of internal heat generation
$t_q$	fluid temperature in presence of wall heat transfer
$t_q^*$	entrance region temperature in presence of wall heat transfer
$t_T$	fluid temperature for uniform wall temperature
$t_T^*$	entrance region temperature for uniform wall temperature
$t_{w, m}$	temperature equal to arithmetic average of wall temperatures, $(t_{w, 1} + t_{w, 2})/2$
$U$	velocity taken as uniform over duct cross section
$u$	local fluid velocity
$W_m(\bar{y}, \eta_m)$	eigenfunction for parallel-plate channel with wall heat transfer and $Q = 0$
$x$	longitudinal coordinate
$Y_m(\bar{y}, \eta_m, e)$	even eigenfunction for parallel-plate channel with wall heat transfer and $Q = 0$
$Y_n^*(\bar{y}, \theta_n)$	eigenfunction for parallel-plate channel with prescribed wall temperature
$y$	transverse coordinate
$\bar{y}$	dimensionless transverse coordinate, $y/a$
$Z_m(\bar{y}, \eta_m, o)$	odd eigenfunction for parallel-plate channel with wall heat transfer and $Q = 0$

$Z_n^*(\bar{y}, \mu_n)$	odd eigenfunction for parallel-plate channel with uniform unequal wall temperatures and $Q \neq 0$
$\beta_j$	eigenvalues for tube flow with wall heat transfer and $Q = 0$
$\Gamma_n(\bar{y}, \lambda_n)$	eigenfunction for parallel-plate channel with uniform unequal wall temperatures and $Q = 0$
$\gamma_k$	eigenvalues of eq. (32)
$\epsilon$	wall heat-flux ratio, $q_{w, 2}/q_{w, 1}$
$\zeta$	dimensionless longitudinal distance, $4(x/D)/RePr$
$\eta_m$	eigenvalues of eq. (68)
$\eta_{m, e}$	eigenvalues of eq. (73)
$\eta_{m, o}$	eigenvalues of eq. (74)
$\theta_n$	eigenvalues of eq. (108)
$\kappa$	thermal conductivity of fluid
$\lambda_n$	eigenvalues of eq. (119)
$\mu$	fluid viscosity
$\mu_n$	eigenvalues of eq. (124)
$\xi$	dimensionless longitudinal distance, $4(x/2a)/RePr$
$\rho$	fluid density

Subscripts:

b	bulk condition of fluid
d	fully developed region
e	entrance, $x = 0$
w	wall condition
1	upper wall of parallel-plate channel, $y = a$
2	lower wall of parallel-plate channel, $y = -a$

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